General Theory of Nuclear Scattering — the Feshbach-Kerman-Koonin approach

by

Goran Arbanas

B.S., Massachusetts Institute of Technology (1990)

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1995

© Massachusetts Institute of Technology 1995. All rights reserved.

Author .....
Department of Physics
May 17, 1995

Certified by .....
Arthur K. Kerman
Professor
Thesis Supervisor

Accepted by .....
George F. Koster
Chairman, Departmental Committee on Graduate Students

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY

JUN 26 1995

LIBRARIES

ARCHIVES

General Theory of Nuclear Scattering — the Feshbach-Kerman-Koonin approach

by

Goran Arbanas

Submitted to the Department of Physics
on May 17, 1995, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

We extend the Feshbach-Kerman-Koonin theory [1] of low and medium energy (10 – 200 MeV) nucleon-nucleus scattering to describe multi-particle emission processes. The original FKK considered processes with only a single outgoing particle. However, multi-particle emission processes become significant at energies as low as 50 MeV and grow in importance at higher energies.

Besides augmenting the Feshbach-Kerman-Koonin theory by adding a description of multi-particle processes, we have improved the theory at the low energy end (10 – 30 MeV) by uncovering a new class of multistep compound processes. We have developed a formalism within the Feshbach-Kerman-Koonin framework to describe these processes while retaining the elegant structure of the Feshbach-Kerman-Koonin theory. We use this formalism to analyze the 14 MeV ^{93}Nb(n, n') reaction and find an improved agreement between the theoretical prediction and the experimental data.

Thesis Supervisor: Arthur K. Kerman
Title: Professor

Acknowledgments

I am deeply grateful to Prof. Kerman for his patient guidance over many years. I am privileged to say that he has not only deepened my knowledge and understanding of physics, but has helped me become a better person while doing so. He has been a very gentle person who genuinely cares about all people around him. During the many years over which we discussed physics he has never made me feel inferior because of lack of knowledge.

Of course, my parents were always there in the case of need. Their support meant much more than I was willing to acknowledge at the time I received it. I found a great source of inspiration in the interest my Father showed for different aspects of my research. He followed some of the topics I worked on with great curiosity and his enthusiasm carried over to me.

I am also very lucky to have met a wonderful person Yuko Hori whose lively spirit was a constant source of laughter and joy. She was my secret source of motivation.

Over the past two years Mark Chadwick of Lawrence Livermore National Laboratory has showed a tremendous amount of patience and encouragement in guiding me through the intricacies of the Feshbach-Kerman-Koonin theory and the details of its computation. I am also indebted to him for his detailed comments on the initial drafts of this thesis as well as his guidance in doing all of the numerical calculations performed in this thesis. Also, thanks to Roger White and Frank Dietrich for their support and constant encouragement.

It was my pleasure to spend many happy moments during the past year with JEN associates: Dan Gilden, Dan Valentine, and Phil Bertoni, from whom I have learned a great deal about the business world while having a really good time. I am particularly indebted to Dan Valentine for his patient guidance through the intricacies of VAX Fortran and sophisticated statistics.

Over the years I have had many comforting conversations with Dapeng Xu, some of which have taken place on our memorable drive across the country in the summer of '93.

There are many tennis partners with whom I had the opportunity to share the excitement and passion for tennis but with no one have I spent more time on and off the tennis courts than with Frederic Silvestre. Also I spent many pleasant moments playing tennis with Joel Freundlich to who I am indebted for challenging me to raise the level of my play.

Contents

List of Figures

Chapter 1

Introduction

1.1 Overview of Low to Medium Energy Nuclear Reactions

In this thesis we construct a quantum statistical theory of low to medium energy (10-200 MeV projectile energy) nucleon-nucleus scattering. The theory we present is based on the Feshbach-Kerman-Koonin theory (FKK) [1] of nuclear scattering, and shares its crucial statistical assumptions and other significant features. Just like FKK, our theory applies to pre-equilibrium processes that are nevertheless statistical in nature.

Both theories are based on statistical assumptions and thus apply primarily to nuclei with large atomic numbers. However, the theory has been successfully applied to nuclei as light as carbon and oxygen. There is no upper limit of the atomic numbers to which the theory applies; in fact, the heavier the nucleus, the more accurate the approximations.

In order to describe how pre-equilibrium processes described by our theory relate to other kinds of well known processes important in this energy range we sketch a schematic cross section in this energy range in Fig 1-1.

By looking at different parts of the emission spectrum we see that low to medium energy nuclear processes may be classified into *direct*, *pre-equilibrium*, and *compound*

processes. Even though the theory presented in this thesis applies to pre-equilibrium processes, we give here an overview of direct and compound processes as well.

Direct processes occur at low energy transfer when the projectile excites distinct, low-lying nuclear states of the target nucleus. These processes manifest themselves in isolated peaks near the high end of the emission spectrum, each peak corresponding to a particular low lying nuclear state excited in a reaction. Direct processes are mostly single-step processes; there is only one interaction between the projectile and the target nucleus. The reaction time is thus relatively short, on the order of the time it takes the projectile to traverse the diameter of the nucleus (approximately $10^{-22}s$). The cross-section is strongly forward peaked and shows an oscillatory behavior from which the quantum numbers of the excited states may be deduced. Direct processes can be described by a Distorted Wave Born Approximation (DWBA) or by the coupled channels formalism [27, 28].

Compound processes, in contrast to direct, occur in the smooth, low energy part of the emission spectrum. They require many steps and consequently have much longer interaction time (approximately $10^{-18}s$). In a compound process, the projectile is captured by the target nucleus and equilibrates with the other nucleons. Subsequently, one or more nucleons are emitted via an evaporation-like process. Because the projectile becomes indistinguishable from the other nucleons, all information about the initial direction is lost and the distribution of emitted particles is spherically symmetric [1]. Hauser- Feshbach theory is most commonly used to describe these processes [34].

Filling the gap between the extremes of direct and compound processes are the so called *pre-equilibrium*, *pre-compound*, or *multi-step* processes. Just as their name suggests, the final particle is emitted before the equilibrium between the projectile and the nucleus has been established.

Pre-equilibrium processes may be further divided into two distinct categories [1]: multi-step direct (MSD) and multi-step compound (MSC) processes. MSD and di-

[1]It is spherically symmetric for nucleon or proton induced reactions which are considered in this thesis. However, for heavy ion compound processes it is only symmetric about 90°

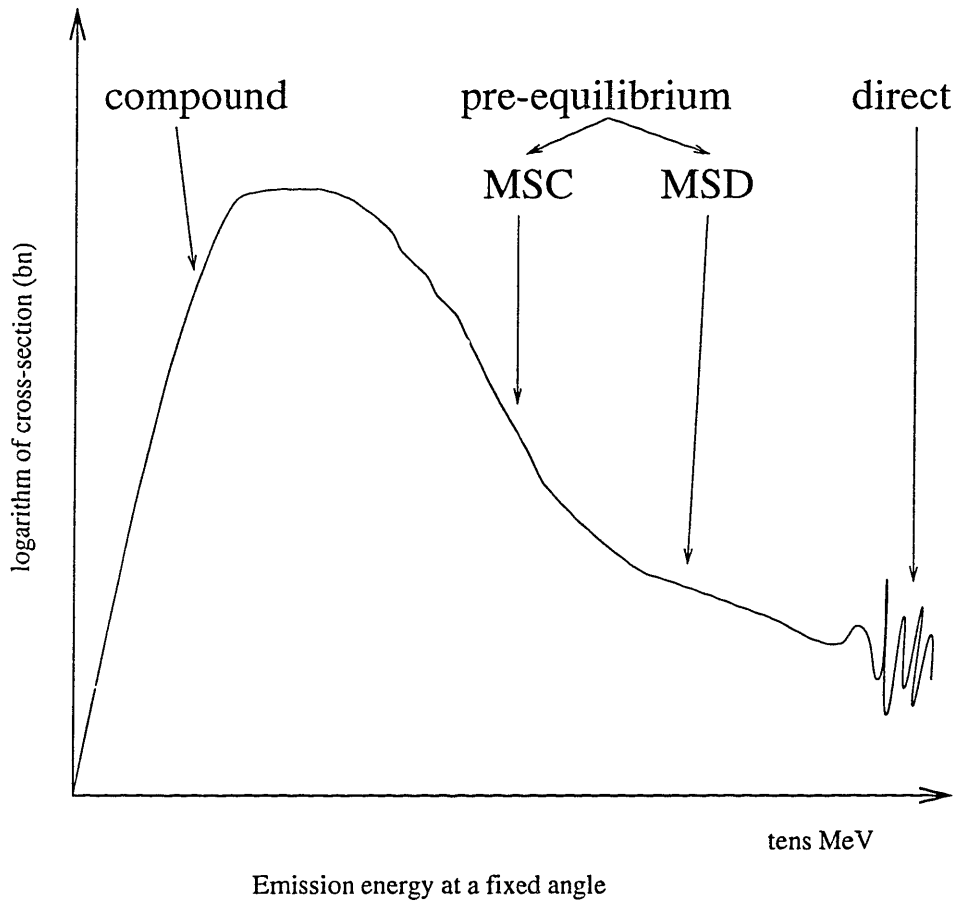

Figure 1-1: This figure shows crude features of cross-section vs. energy behavior for the medium-low energy nucleon-nucleus scattering. Cross section is measured at a fixed angle and is plotted on logarithmic scale. Different processes and regions in the cross-section where they make a dominant contribution are noted. The energy scale is equal to the projectile's energy.

rect processes are closely related and have similar properties. Likewise MSC and compound processes are similar.

A crucial difference between MSD and direct processes is that the energy transferred to the nucleus in MSD reactions is larger than that in direct reactions. Consequently, the density of residual nuclear states is high enough for statistical treatment to be appropriate. A direct consequence of applying statistical methods to MSD processes is a smoothing of the cross-section which is now averaged over many nuclear states. By averaging we lose information about specific nuclear states and the oscillatory behavior prominent in direct reactions disappears. However, the cross section still remains forward peaked, albeit somewhat less than in the case of direct scattering.

On the emission energy spectrum, MSD processes are important at energies immediately below those of direct processes. As we move toward lower energies, MSC processes grow in importance. There may be a substantial overlap between emission spectra of MSD and MSC. As we go to even lower energies, the MSC contribution becomes significant but is overpowered by pure compound processes at the lowest energies.

In MSC processes the projectile is captured by the nucleus, as in compound reactions. However, in contrast to compound reactions, emission in MSC reactions can occur long before equilibrium has been established. Since equilibrium is not established at the time of emission, the emitted particle still carries some information about the initial direction of the projectile and consequently the angular distribution is symmetrical about 90°. In addition, the outgoing particle has somewhat higher energy than one emitted from an equilibrated nucleus. Like MSD, the treatments of MSC processes employ statistical methods and assumptions.

FKK assumes that different steps of a multistep process are statistically independent. There exist two other quantum mechanical theories of pre-equilibrium, multistep processes: Tamura-Udagawa-Lenske (TUL) (1982) [29], and Nishioka-Weidenmuller-Yoshida (NWY) (1988) [30] which retain correlations between successive steps. However, these theories do not show improved agreement with experimental data. Thus

we see no reason to introduce the additional complexity of these theories.

Even though this thesis is based on FKK in its entirety, we describe the main features of the other two theories and their common ground with the FKK.

All three theories share the intuitive picture of the projectile interacting with the target nucleus creating an additional particle-hole pair in each consecutive step. Furthermore, all three theories are statistical in nature but differ in the statistical assumptions and approximations they employ. Finally, all three theories derive identical expressions for the single step scattering, and comparison of the numerical results of the three theories has shown that all three theories give similar predictions.

Furthermore, the final form of the cross-section in the FKK theory makes it the most attractive for generalizing to processes we consider in this thesis. This is because the FKK makes as many statistical assumptions as are allowed by the physical processes under consideration, ensuring that the final form of the cross-section is as simple as possible. The simple structure of the FKK is preserved in the cross-sections obtained for the novel processes considered in this thesis.

1.2 Further Classification of Multistep Compound and Multistep Direct Processes

MSC processes are subdivided into four distinct categories, three of which were discovered as a part of this thesis research and will be described in great detail in Chapter 4. MSD processes, on the other hand, may be classified by the number of continuum particles in the final state. These processes will be discussed in Chapters 2 and 3.

FKK, NWY, and TUL describe only those MSD processes which have a single particle in the continuum. Several other authors have described two-particle final state scattering in the context of the FKK formalism; most notably, Ciangaru extended the FKK formalism in a rigorous way [18] while Chadwick *et al.* applied phase space arguments [33]. In this thesis we extend the formalism to include an arbitrary number of continuum particles.

1.3 Practical Applications of the Feshbach-Kerman-Koonin Calculations

FKK calculations of cross-sections have been actively applied in several different areas.

In oncological studies, cross-sections of neutron scattering on carbon and oxygen have been used in Monte-Carlo simulations of a neutron beam passing through a human body. In nuclear reactor design, cross-sections of low-energy neutrons on lead have been used to determine shielding requirements of reactors. Finally, proposals for accelerator production of tritium (APT) and accelerator transmutation of waste (ATW) rely on calculated cross-sections of low energy neutrons.

1.4 Overview of This Work

In Section 2.1 we will review FKK methods in order to present the FKK cross-section for the MSD processes. This will be used as a springboard to derive cross-section for two-particle MSD emission in Section 2.2, which will then be generalized to a many particle cross-section in Section 2.3.

In Chapter 3 we will derive an FKK cross-section for generalized MSC processes, including the infamous $P \rightarrow Q$ transitions.

Disagreement has long existed over whether the multistep direct cross section is a convolution of DWBA matrix elements as advocated by Feshbach [16], or a convolution of modified DWBA elements (intermediate steps) and a DWBA element (the final step). In Chapter 4 we present an algorithm by which intermediate step amplitude can be determined from an experimentally measured cross-section. Comparison of the intermediate step amplitude and the final step amplitude obtained by this algorithm reveals whether the intermediate steps are DWBA or not.

In order to extend the FKK formalism to even higher energies, in Chapter 5 we explore connections between multistep t-matrix scattering formalism (a formalism known to be valid at several hundred MeV) and the FKK theory.

Chapter 2

General Multistep Direct (MSD) Processes

2.1 The original Feshbach-Kerman-Koonin Theory of MSD Reactions

2.1.1 Introduction

The Feshbach-Kerman-Koonin (FKK) statistical theory of multi-step direct (MSD) nuclear scattering derives a quantal expression for the pre-equilibrium, multi-step nuclear scattering cross section. Because FKK is a theory based on statistical assumptions, it derives an averaged cross section which describes the global structure of a measured cross sections but not its microstructure or contributions from non-statistical processes such as isolated doorway states. However, because of the random phase approximation, the cross section for non-statistical processes can be incoherently added to that of statistical processes[1]. The most recent comprehensive review of the Feshbach-Kerman-Koonin multistep direct reaction theory and its applications can be found in [38].

[1]This is a direct consequence of the random phase approximation which will be described and used extensively later on.

2.1.2 Complexity of States and the Chaining Hypothesis

According to FKK, a multistep process proceeds in a sequence of two-body interactions between the projectile and individual nucleons inside the target nucleus. In each interaction, or step, the projectile excites one of the bound nucleons and by doing so creates an particle-hole pair.

Depending on the amount of the energy deposited by the projectile in the target, the projectile either remains an unbound continuum particle or becomes a bound particle inside the target. If the projectile remains unbound throughout a multistep reaction, the reaction is called a multi-step direct (MSD) reaction. If the projectile becomes bound, even for only a single step of the process, the reaction is called multi-step compound (MSC). In MSC processes, information about the direction of incidence of the projectile is lost when the projectile becomes bound; thus the corresponding angular emission distribution is symmetric about 90°[2].

Because the particle-hole density of states increases rapidly with the number of excited particles and holes [3] larger number of steps in a given reaction yields more complex nuclear states. Using this observation FKK classifies all possible nuclear configurations based on their complexity, where complexity is measured by the number of particle-hole pairs. Since a particle-hole pair is created in every step, the number of such pairs in a given configuration is equal to the number of steps it takes to reach that configuration.

To illustrate, consider a projectile incident on a nuclear target. Since this theory is restricted to two-body interactions, the projectile interacts with a single nucleon in its first step, exciting it above the Fermi level. The struck nucleon leaves a nuclear hole in its original place. The energy lost by the projectile in the first step equals the energy received by the struck nucleon so that *energy is conserved* in each and every step.

After the first interaction in an MSD process, the nuclear configuration is designated 1p1h because it includes exactly one particle-hole pair. If the projectile becomes

[2]In fact, for all computational purposes, the MSC cross section is spherically symmetric.
[3]The Williams expression for the density of states is given in Equation (2.37).

bound after the first step, as is the case in MSC processes, the resulting configuration would be 2p1h.

In the FKK theory, the space of nuclear wave-functions spanning all configurations in which the projectile remains unbound at all stages, is labeled P, and the corresponding space for configurations in which the projectile becomes bound is labeled Q. These two classes of processes are treated separately. The complexity-based classification mentioned earlier is performed for P and Q processes separately: P space is divided into subspaces of different complexities (P_1, P_2, ..., P_μ, ... where μ is the number of particle-hole pairs) and Q space is separately divided in the same fashion. The situation in Q-subspaces is slightly different since the captured projectile brings an additional particle to the nuclear configuration. Therefore, subspace Q_ν has $(\nu + 1)$ particles and ν holes.

Figure 2-1 shows the P and Q-chain subspaces introduced by FKK theory. Different processes are represented by arrows connecting different subspaces. In principle, processes can go in the direction of either increasing or decreasing complexity. Even though it is unlikely that a process will make a step in the direction of decreasing complexity[4] , allowing this possibility is important in order to be able to factorize propagators as shown in Appendix A. However, the phase space of a more complex subspace is much larger and therefore the likelihood for a process to proceed to more complex states is much larger. In the literature this is sometimes referred as the 'never come back' hypothesis and it somewhat simplifies the expression for the cross section.

Emission to a final state may occur from any given stage as indicated by dashed lines in Fig 2-1. In case of a Q-chain emission, the particle has to pass through one of the neighboring P-subspaces before going into the final state.

In a single step, the number of particle-hole pairs can change by at most one. In terms of subspace labels this means that from an arbitrary subspace P_μ only $P_{\mu+1}$, P_μ and $P_{\mu-1}$ can be reached in a single step. (The same is true for the Q subspaces.)

[4]If the states are so complex that an equilibrium with many excited particles and holes has been established than the likelihood of of going to a less complex stage is significant.

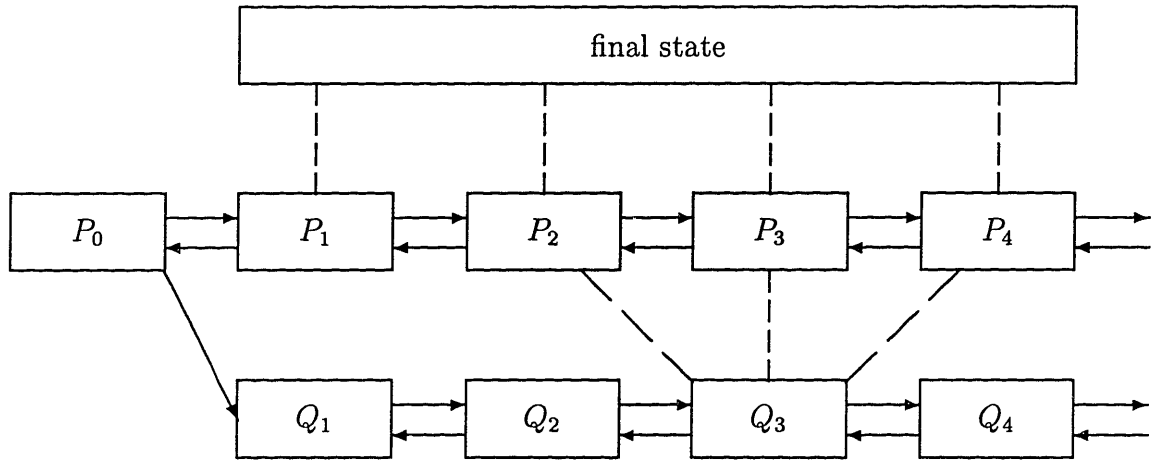

Figure 2-1: Processes considered in the original FKK theory are indicated by arrows. Q chain could be entered in the first step only (i.e. $P_0 \rightarrow Q_1$). For the projectile to be emitted from the Q chain to the final state, it has to go through a P space (as indicated by dashed lines going from Q_3 to P_2, P_3, and P_4). Similarly, an emission to the final state from a particular P_n subspace also goes through its neighboring subspaces P_{n-1}, P_n, and P_{n+1}

These three possibilities summarize the chaining hypothesis.

2.1.3 Partition of the $\mathcal{T}_{fi}$-matrix into MSC and MSD parts

As described earlier, FKK divides multi step processes into two classes: multi-step direct (MSD) and multi-step compound (MSC) processes, which are treated separately due to their different statistical properties. The cross sections for these two classes are added at the end in order to obtain the total multi-step cross section. Formally this is done by separating the total transition matrix element into MSD and MSC parts. Upon squaring and averaging the the total $\mathcal{T}_{fi}$ matrix, the cross terms vanish, resulting in the cross section being a sum of MSD and MSC individual cross sections.

$$\mathcal{T}_{fi} = \mathcal{T}_{fi}^{MSD} + \mathcal{T}_{fi}^{MSC} \tag{2.1}$$

The separation of processes in the way just described is a fundamental feature underlying the original and all subsequent extensions of the FKK theory. The difference, however, between the original FKK and this work is that in this work many more processes are included in both $\mathcal{T}_{fi}^{MSD}$ and $\mathcal{T}_{fi}^{MSC}$. It is a goal of this thesis

to attempt to include all possible multi-step processes and thus to derive the most general FKK cross section. The novel MSC processes we introduce in Chapter 3 are the so-called P to Q transitions, and the new MSD processes described in Sections 2.2 and 2.3 are the multi-particle emission processes.

Omission of those processes by FKK does not diminish its significance for two reasons: FKK does give a correct cross section for the one particle final state processes they considered, and it gives a clear prescription on how to modify the total cross section for any additional multi-step processes (simply add their cross section to the total cross section). The later feature is a consequence of the main statistical assumption of the theory, namely the random phase hypothesis (RPA), which is used at crucial stages in the development of the formalism and is in a great way responsible for the simple structure of the final cross sections.

A useful feature of the FKK formalism is that it remains valid whether or not the complete set of multi-step processes has been included in calculation of the cross section. Therefore many authors have considered only those multi-step processes that are judged to contribute the most significant fraction of the cross section a particular energy. For example, Ciangaru [18] applied the FKK formalism to derive a two-particle final state cross section. To do so, he considered only those multi-step processes in which knock-out occurs at the first step, followed by multi-step direct scattering of either or both continuum particles. This set of multi-step processes can be further narrowed to processes in which the projectile is emitted immediately after the knock-out process and only the knocked-out particle undergoes MSD rescattering. The latter process describes the data sufficiently well for small deflection angles [20].

To our knowledge, this thesis represents the first attempt to systematically include all multi-step processes in the total FKK cross section. In doing so we have discovered a new sub-class of MSC processes overlooked in previous analyses (already published [21]). Also, we have obtained an expression for the many-particle final state cross section and consequently disproved the belief expressed by Ciangaru [18] that the FKK formalism cannot be used to describe processes with more than two particles in the final state.

2.1.4 Multistep Direct Formalism

In this section we will review the derivation of the cross section for the multistep direct processes with only one particle in the continuum. Even though these processes were analyzed in the original FKK paper [1], we will repeat the derivation in order to illustrate the formalism that will be used when deriving the cross section for MSD processes with more than one particle in the continuum.

One starts with the equation for the averaged P-space wavefunction $P\Psi$:

$$(E - H_{\rm opt})P\Psi = 0 \tag{2.2}$$

where $H_{\rm opt}$ is the multichannel optical model Hamiltonian. One then proceeds by separating $H_{\rm opt}$ into a diagonal component and a coupling interaction v with respect to the set of the target nuclear states Ψ_α:

$$\begin{aligned} H_{\rm opt} &= \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| H_{\rm opt} \sum_\beta |\Psi_\beta\rangle\langle\Psi_\beta| && (2.3)\\ &= \sum_\alpha |\Psi_\alpha\rangle H_\alpha \langle\Psi_\alpha| + \sum_{\alpha\neq\beta} |\Psi_\alpha\rangle\langle\Psi_\alpha| H_{\rm opt} \sum_\beta |\Psi_\beta\rangle\langle\Psi_\beta| && (2.4)\\ &= H^{(D)} + v && (2.5) \end{aligned}$$

so that the solution for $P\Psi$ can be written as

$$P|\Psi_i^{(+)}\rangle = |\phi_i^{(+)}\rangle + \frac{1}{E^{(+)} - H_{\rm opt}} v|\phi_i^{(+)}\rangle \tag{2.6}$$

where $(E - H^{(D)})\phi_i = 0$. The $\mathcal{T}_{fi}^{MSD}$-matrix is then

$$\begin{aligned} \mathcal{T}_{fi}^{MSD} &\equiv \langle\phi_f^{(-)}| v P |\Psi_i^{(+)}\rangle && (2.7)\\ &= v_{fi} + \langle\phi_f^{(-)}| v \frac{1}{E^{(+)} - H_{\rm opt}} v|\phi_i^{(+)}\rangle && (2.8) \end{aligned}$$

where $v_{fi} = \langle \phi_f^{(-)} | v | \phi_i^{(+)} \rangle$. Next, the FKK separates the $\mathcal{T}_{fi}^{(MSD)}$ into contributions coming from different stages:

$$\mathcal{T}_{fi}^{MSD} = v_{fi} + \sum_{\mu} \mathcal{T}_{fi}^{(\mu)} \tag{2.9}$$

where the label μ represents the contribution from the P_μ-stage:

$$\mathcal{T}_{fi}^{(\mu)} = \langle \phi_f^{(-)} | v_{P_f P_\mu} \frac{1}{E - H_{\rm opt}} v_{P_1 P_i} | \phi_i^{(+)} \rangle \tag{2.10}$$

P_μ is a projection operator that projects onto the $\mu^{\rm th}$ subspace in the P-chain, i.e. the subspace of nuclear wavefunctions with μ particle-hole pairs. Obviously, $\sum_\mu P_\mu = P$. Since the wavefunctions from different subspaces are orthonormal, the projection operators are orthogonal:

$$P_\mu P_\nu \; = \; \delta_{\mu\nu} P\mu \tag{2.11}$$

In terms of the projection operators, the coupling potential can be written as

$$v \; = \; \sum_{\mu\nu} v_{P_\mu P_\nu} \qquad v_{P_\mu P_\nu} = P_\mu v P_\nu \tag{2.12}$$

Next we employ the chaining hypothesis (i.e. $v_{\mu\nu} = 0$ when $|\mu - \nu| > 1$) to factorize the propagator in $\mathcal{T}_{fi}^{(\mu)}$, as shown in appendix A:

$$\begin{aligned} P_\mu \frac{1}{E^{(+)} - H_{\rm opt}} P_1 \; &= \; G_{P_\mu} v_{P_\mu P_{\mu-1}} P_{\mu-1} \frac{1}{E^{(+)} - H_{\rm opt}} P_1 \\ &= \; G_{P_\mu} v_{P_\mu P_{\mu-1}} G_{P_{\mu-1}} v_{P_{\mu-1} P_{\mu-2}} \cdots G_{P_2} v_{P_2 P_1} G_{P_1} P_1 \end{aligned} \tag{2.13}$$

so that

$$\mathcal{T}_{fi}^{(\mu)} = \langle \phi_f^{(-)} | v_{P_f P_\mu} G_{P_\mu} v_{P_\mu P_{\mu-1}} G_{P_{\mu-1}} v_{P_{\mu-1} P_{\mu-2}} \cdots G_{P_2} v_{P_2 P_1} G_{P_1} P_1 v_{P_1 P_i} | \phi_i^{(+)} \rangle \tag{2.14}$$

where

$$G_{P_\mu} = \frac{1}{E^{(+)} - H^{(D)}_{P_\mu} - v_{P_\mu P_{\mu+1}} G_{P_{\mu+1}} v_{P_{\mu+1} P_\mu}} \tag{2.15}$$

Next, we define the eigenfunctions of the Hamiltonian $H^{(D)}_{P_\mu} - v_{P_\mu P_{\mu+1}} G_{P_{\mu+1}} v_{P_{\mu+1} P_\mu}$ in the above propagator:

$$(H^{(D)}_{P_\mu} + v_{P_\mu P_{\mu+1}} G_{P_{\mu+1}} v_{P_{\mu+1} P_\mu}) \Psi^{(+)}_{P_\mu \alpha} = (\frac{\hbar}{2m} k^2_{P_\mu} + \mathcal{E}_{P_\mu \alpha}) \Psi^{(+)}_{P_\mu \alpha} \tag{2.16}$$

Because the above Hamiltonian has an imaginary part, the eigenfunctions which are orthonormal to those defined above satisfy

$$(H^{(D)}_{P_\mu} + v_{P_\mu P_{\mu+1}} G_{P_{\mu+1}} v_{P_{\mu+1} P_\mu})^\dagger \tilde{\Psi}^{(+)}_{P_\mu \alpha} = (\frac{\hbar}{2m} k^2_{P_\mu} + \mathcal{E}_{P_\mu \alpha}) \tilde{\Psi}^{(+)}_{P_\mu \alpha}. \tag{2.17}$$

The Green's function G_{P_μ} is then expanded in terms of the wavefunctions defined above:

$$G_{P_\mu} = \sum_\alpha \int \frac{d\mathbf{k}_\mu}{(2\pi)^3} \frac{|\Psi^{(+)}_{P_\mu \alpha}\rangle \langle \tilde{\Psi}^{(+)}_{P_\mu \alpha}|}{E^{(+)} - \frac{\hbar^2}{2m} k^2_\mu - \mathcal{E}_{\mu \alpha}} \tag{2.18}$$

The index α labels nuclear states with excitation energy equal to the energy transferred from the projectile to the nucleus in a single step, i.e. energy $U = E - (\hbar/2m)k_i^2$

Throughout our derivation, unlike in the original FKK, we will continue using the orthonormal set of wave functions $|\Psi^{(+)}_{P_\mu \alpha}\rangle \langle \tilde{\Psi}^{(+)}_{P_\mu \alpha}|$ as indicated in the expansion of the above Green's function. In the original derivation of FKK it was assumed that upon energy averaging, the wave function $\langle \tilde{\Psi}^{(+)}_{P_\mu \alpha}|$ may be replaced by its biorthogonal conjugate $\langle \Psi^{(-)}_{P_\mu \alpha}|$. This substitution increases the aesthetic appeal of the final expression for the MSD cross section since it introduces DWBA boundary conditions into the matrix elements, making them all DWBA. Although the need for this substitution was advocated by Feshbach [16] it has not been generally accepted. Until the resolution of this controversy, we continue using the set of wavefunctions indicated in the

expansion of the Green's function without assuming a particular effect of energy averaging on the matrix elements. In either case, the structure of the theory presented here will be the same; one can simply change the matrix elements to DWBA if the substitution turns out to be justified.

We proceed to evaluate the averaged cross section for MSD processes which is given by

$$\frac{d\sigma^{MSD}(\mathbf{k}_f \leftarrow \mathbf{k}_i)}{dE_{k_f} d\Omega_{k_f}} = \frac{2\pi}{\hbar} \frac{m}{\hbar k_i} \rho(\quad , \sum_{\mu} \langle |\mathcal{T}_{fi}^{(\mu)}|^2 \rangle \tag{2.19}$$

The average of the square of the $\mathcal{T}_{fi}^{MSD} = \sum_{\mu} \mathcal{T}_{fi}^{(\mu)}$ is

$$\langle |\mathcal{T}_{fi}^{MSD}|^2 \rangle = \langle \sum_{\mu\nu} \mathcal{T}_{fi}^{(\mu)} \mathcal{T}_{fi}^{(\nu)*} \rangle \tag{2.20}$$

which upon averaging and using the random phase approximation (by which all $\mu \neq \nu$ terms can be set to zero) becomes

$$\langle |\mathcal{T}_{fi}^{MSD}|^2 \rangle = \sum_{\mu} \langle \mathcal{T}_{fi}^{(\mu)} \mathcal{T}_{fi}^{(\mu)*} \rangle \tag{2.21}$$

$$= \sum_{\mu} \langle \langle \phi_f^{(-)} | v_{P_f P_\mu} G_{P_\mu} v_{P_\mu P_{\mu-1}} \cdots v_{P_2 P_1} G_{P_1} v_{P_1 P_i} | \phi_i^{(+)} \rangle$$
$$\langle \phi_i^{(-)} | v^*_{P_i P_1} G^*_{P_1} v^*_{P_1 P_2} \cdots v^*_{P_{\mu-1} P_\mu} G^*_{P_\mu} v^*_{P_\mu P_f} | \phi_f^{(+)} \rangle \rangle \tag{2.22}$$

For the moment we concentrate on the $P_0 \rightarrow P_1 \rightarrow P_2$ part of an MSD process, from which it will be easy to construct the general expression for an arbitrary number of steps. In other words we consider the $v_{P_2 P_1} G_{P_1} v_{P_1 i} |\phi_i^{(+)}\rangle \langle \phi_i^{(-)}| v^*_{i P_1} G^*_{P_1} v^*_{P_1 P_2}$ part of the above equation (2.22). We begin by substituting the Green's functions (2.18) into this part of the transition matrix element:

$$
\begin{aligned}
v_{P_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\phi_i^{(-)}|v^*_{iP_1}G^*_{P_1}v^*_{P_1P_2} &= \sum_{\alpha\alpha'}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}'_1}{(2\pi)^3} \\
&\times\bar{v}_{\beta\alpha}(\mathbf{k}_2,\mathbf{k}_1)\frac{1}{E-(\hbar^2/2m)k_1^2-\mathcal{E}_{1\alpha}+i\eta}\bar{v}_{\alpha i}(\mathbf{k}_1,\mathbf{k}_i) \\
&\times\bar{v}^*_{i\alpha'}(\mathbf{k}_i,\mathbf{k}'_1)\frac{1}{E-(\hbar^2/2m)k_1'^2-\mathcal{E}_{1\alpha'}-i\eta}\bar{v}^*_{\alpha'\beta'}(\mathbf{k}'_1,\mathbf{k}'_2) \quad (2.23)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{v}_{\alpha i}(\mathbf{k}_1,\mathbf{k}_i) &= \langle\tilde{\Psi}^{(+)}_\alpha(\mathbf{k}_1)|v|\phi^{(+)}_i(\mathbf{k}_i)\rangle \\
\bar{v}_{\beta\alpha}(\mathbf{k}_2,\mathbf{k}_1) &= \langle\tilde{\Psi}^{(+)}_\beta(\mathbf{k}_2)|v|\Psi^{(+)}_\alpha(\mathbf{k}_1)\rangle \qquad (2.24)
\end{aligned}
$$

What now follows is a sequence of random phase approximations which will be applied to all integration and summation variables in (2.23), resulting in setting all primed variables to their unprimed counterparts. To start out, only $\alpha=\alpha'$ terms in the sum will contribute and therefore only one summation over α remains. This sum is then cast into an energy integral over $\mathcal{E}_{1\alpha}$

$$
I=\int\frac{\bar{v}_{\beta\alpha}(\mathbf{k}_2,\mathbf{k}_1)\bar{v}_{\alpha i}(\mathbf{k}_1,\mathbf{k}_i)\bar{v}^*_{i\alpha'}(\mathbf{k}_i,\mathbf{k}'_1)\bar{v}^*_{\alpha'\beta'}(\mathbf{k}'_1,\mathbf{k}'_2)}{[E-(\hbar^2/2m)k_1^2-\mathcal{E}_{1\alpha}+i\eta][E-(\hbar^2/2m)k_1'^2-\mathcal{E}_{1\alpha'}-i\eta]}\rho_{1p1h}(\mathcal{E}_{1\alpha})d\mathcal{E}_{1\alpha} \qquad (2.25)
$$

where $\rho_{1p1h}(\mathcal{E}_{1\alpha})$ is the nuclear density of P_1, i.e. the density of 1p1h states. The above integral can be done by assuming that the main contributions come from its singularities. Also one needs to assume that the product of four v's in the numerator does not vary appreciably near the singularities and can therefore be replaced by its average value at the singularity $\mathcal{E}_{1\alpha}=U_1\equiv E-\frac{\hbar^2}{2m}k_1^2$. Averaging of v's over P_1 states with energy U_1 is denoted by putting angle-brackets around them, and replacing index α by 1 to indicate that averaging over all states in P_1 has been performed:

$$I = (2\pi i)\ \rho_{1p1h}(U_1)\ \frac{\langle \bar{v}_{\beta 1}(\mathbf{k}_2,\mathbf{k}_1)\bar{v}_{1i}(\mathbf{k}_1,\mathbf{k}_i)\bar{v}^*_{i1}(\mathbf{k}_i,\mathbf{k}'_1)\bar{v}^*_{1\beta'}(\mathbf{k}'_1,\mathbf{k}'_2)\rangle}{\hbar^2/(2m)(k_1^2-k_1'^2)}. \tag{2.26}$$

The contribution of the last term in the above equation can be divided into its principal part and a δ-function. Assuming slow variation of the product of four v's when $\mathbf{k}'_1$ is near $\mathbf{k}_1$, the contribution to the principal part near its singularity is zero since $(k_1^2-k_1'^2)$ changes sign at $k_1^2 = k_1'^2$. Also, when $\mathbf{k}'_1$ is appreciably different from $\mathbf{k}_1$ the integrand vanishes by the random phase hypothesis, making the principal value of the integral zero. The δ-function contribution is equal to

$$I = 2\pi^2\rho_{1p1h}(U_1)\langle \bar{v}_{\beta 1}(\mathbf{k}_2,\mathbf{k}_1)\bar{v}_{1i}(\mathbf{k}_1,\mathbf{k}_i)\bar{v}^*_{i1}(\mathbf{k}_i,\mathbf{k}'_1)\bar{v}^*_{1\beta'}(\mathbf{k}'_1,\mathbf{k}'_2)\rangle\ \delta\left(\frac{\hbar^2}{2m}(k_1^2-k_1'^2)\right) \tag{2.27}$$

which upon inserting into (2.23) gives

$$\begin{aligned} v_{P_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\phi_i^{(-)}|v^*_{iP_1}G^*_{P_1}v^*_{P_1P_2} \;&=\; \int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}'_1}{(2\pi)^3}2\pi^2\rho_{1p1h}(U_1)\ \delta\left(\frac{\hbar^2}{2m}(k_1^2-k_1'^2)\right) \\ &\times\langle \bar{v}_{\beta 1}(\mathbf{k}_2,\mathbf{k}_1)\bar{v}_{1i}(\mathbf{k}_1,\mathbf{k}_i)\bar{v}^*_{i1}(\mathbf{k}_i,\mathbf{k}'_1)\bar{v}^*_{1\beta'}(\mathbf{k}'_1,\mathbf{k}'_2)\rangle \end{aligned} \tag{2.28}$$

Next, by employing the definition of the density of states,

$$\rho(k_1) = \int\frac{d\mathbf{k}'_1}{(2\pi)^3}\ \delta\left(\frac{\hbar^2}{2m}(k_1^2-k_1'^2)\right), \tag{2.29}$$

and the random phase hypothesis, which in this case says that only $\mathbf{k}'_1 = \mathbf{k}_1$ gives a non-zero contribution, we obtain:

$$\begin{aligned} v_{P_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\phi_i^{(-)}|v^*_{iP_1}G^*_{P_1}v^*_{P_1P_2} \;&=\; \int\frac{d\mathbf{k}_1}{(2\pi)^3}2\pi^2\rho_{1p1h}(U_1)\rho(k_1) \\ &\times\langle \bar{v}_{\beta 1}(\mathbf{k}_2,\mathbf{k}_1)|\bar{v}_{1i}(\mathbf{k}_1,\mathbf{k}_i)|^2\bar{v}^*_{1\beta'}(\mathbf{k}_1,\mathbf{k}'_2)\rangle \end{aligned} \tag{2.30}$$

Averaging of $\langle \bar{v}_{\beta 1}(\mathbf{k}_2, \mathbf{k}_1)|\bar{v}_{1i}(\mathbf{k}_1, \mathbf{k}_i)|^2 \bar{v}^*_{1\beta'}(\mathbf{k}_1, \mathbf{k}'_2)\rangle$ can be simplified if one notices that in the next step the averaging over states in P_2 will have to be performed. Since there are many more states in P_2 than in P_1, the averaging over P_1 space will affect only the inside term, i.e. $|\bar{v}_{1i}(\mathbf{k}_1, \mathbf{k}_i)|^2$. Consequently, the averaging over nuclear wavefunctions in P_1 can be performed on the $|\bar{v}_{1i}(\mathbf{k}_1, \mathbf{k}_i)|^2$ without regard to the $\bar{v}_{\beta 1}$ to the left and $\bar{v}^*_{1\beta'}$ to the right of it. Identical arguments will be used for for averaging in P_2 and all higher P subspaces.

The process is repeated in a completely analogous way for summations over β, β', etc., and integrations over $\mathbf{k}_2$ and $\mathbf{k}'_2$, ... $\mathbf{k}_\mu$ and $\mathbf{k}'_\mu$, after which the projectile is emitted to the final state. However, because it is experimentally impossible to isolate a single final state the additional averaging over final states has to be performed. Since the final state can be in any of the P_m subspaces connected to P_μ by the chaining hypothesis, the averaging is to be done over the $P_{\mu-1}$, P_μ, and $P_{\mu+1}$ components of the final state. The random phase hypothesis is employed in summing over the P_m, and the MSD cross section is obtained:

$$
\begin{aligned}
|\mathcal{T}^{(\mu)}_{fi}|^2 \;=\; & \sum_{m=\mu-1}^{m=\mu+1} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d\mathbf{k}_\mu}{(2\pi)^3} \left[\frac{1}{2\pi^2\,\rho(k_f)}\right] \left[\frac{d^2 w^{(1)}_{m,\mu}(\mathbf{k}_f \leftarrow \mathbf{k}_\mu)}{dU_f d\Omega_f}\right] \\
& \times \left[\frac{d^2 \tilde{w}^{(1)}_{\mu,\mu-1}(\mathbf{k}_\mu \leftarrow \mathbf{k}_{\mu-1})}{dU_\mu d\Omega_\mu}\right] \cdots \left[\frac{d^2 \tilde{w}^{(1)}_{1,i}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right], \qquad (2.31)
\end{aligned}
$$

where

$$
\left[\frac{d^2 w^{(1)}_{m,\mu}(\mathbf{k}_f \leftarrow \mathbf{k}_\mu)}{dU_f d\Omega_f}\right] = 2\pi^2\, \rho_m(U_f)\rho(k_f)\langle|\langle\Psi^{(-)}_m(\mathbf{k}_f)|v|\Psi^{(+)}_\mu(\mathbf{k}_\mu)\rangle|^2\rangle \quad (2.32)
$$

$$
\left[\frac{d^2 \tilde{w}^{(1)}_{\nu,\nu-1}(\mathbf{k}_\nu \leftarrow \mathbf{k}_{\nu-1})}{dU_\nu d\Omega_\nu}\right] = 2\pi^2\, \rho_\nu(U_\nu)\rho(k_\nu)\langle|\langle\tilde{\Psi}^{(+)}_\nu(\mathbf{k}_\nu)|v|\Psi^{(+)}_{\nu-1}(\mathbf{k}_{\nu-1})\rangle|^2\rangle \quad (2.33)
$$

As indicated in the above equation, only the final term, $\langle|\langle\Psi^{(-)}_m(\mathbf{k}_f)|v|\Psi^{(+)}_\mu(\mathbf{k}_\mu)\rangle|^2\rangle$, has DWBA boundary conditions; all other terms have matrix elements with boundary conditions dictated by those of the wavefunctions used in the expansion of the intermediate Green's function (2.18). The fact that these are not DWBA boundary

conditions is indicated by placing " ~ " over w. Evaluating the cross section is done by inserting the above expression for the $|\mathcal{T}_{fi}^{\mu}|^2$ into (2.19).

In the equation that follows, the single step amplitudes have been converted to cross sections, and the integration over momenta has been changed to integration over energy and angle. Algebraic details of these two conversions are described in Appendix B.

Furthermore, in the following equation we have employed the *spectator hypothesis*, which says that a single-step reaction at any given stage is independent of the particle-hole configuration at that stage. This is a very reliable approximation for relatively heavy nuclei undergoing processes including a small number of steps, which is usually the case. The spectator hypothesis will be discussed in more detail in Subsection 2.2.3.

Finally, we have employed the so called *never-come-back* hypothesis which states that the final state is most likely to be in the $P_{\mu+1}$-subspace, and consequently the sum over m, in Equation (2.31) is approximated by its $m = \mu + 1$ contribution.

Substitution of (2.31) into (2.19) together with the above approximations yields the contribution of μ-step processes to the total differential cross section:

$$\frac{d^2\sigma^{(\mu)}(E_f, \Omega_f \leftarrow E_0, \Omega_0)}{d\Omega_f dE_f} = \left(\frac{m}{4\pi^2\hbar^2}\right)^{\mu} \int dE_1 E_1 d\Omega_1 \cdots dE_\mu E_\mu d\Omega_\mu \times \left[\frac{d^2\sigma^{(1)}(\mathbf{k}_f \leftarrow \mathbf{k}_\mu)}{dU_f d\Omega_f}\right] \left[\frac{d^2\tilde{\sigma}^{(1)}(\mathbf{k}_\mu \leftarrow \mathbf{k}_{\mu-1})}{dU_\mu d\Omega_\mu}\right] \cdots \left[\frac{d^2\tilde{\sigma}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right] \quad (2.34)$$

The single step DWBA cross section is given by

$$\frac{d^2\sigma^{(1)}(E, \Omega \leftarrow E_0, \Omega_0)}{d\Omega dE} = \sum_l (2l+1)\rho(1p, 1h, E_0 - E, l)\left\langle \left[\frac{d\sigma(E, \Omega \leftarrow E_0, \Omega_0)}{d\Omega}\right]_l^{DWBA} \right\rangle. \quad (2.35)$$

The difference between σ and $\tilde{\sigma}$ parallels that between w and $\tilde{w}$ in (2.32) and (2.33). We use the Williams formula [24] for the density of states of a particle-hole configuration, which uses the equidistant energy level approximation:

$$\rho(1p,1h,E_0-E,l) = \omega(p,h,E)\ R_n(l) \tag{2.36}$$

$$\omega(p,h,E) = \frac{g^n}{p!h!(n-1)!}(E-\Delta-A_{ph})^{n-1}, \tag{2.37}$$

where $n = p + h$ and the single particle spacing is $g = A/13$. The Pauli-blocking factor is $A_{ph} = [p^2 + h^2 + p - 3h]/4g$, and the pairing energy corrections (Δ) of Dilg *et al.* [25] are used. A Gaussian angular momentum distribution is assumed:

$$R_n(l) = \frac{2l+1}{2\sqrt{2\pi}\sigma_n^3}\exp\left[-\frac{(l+1/2)^2}{2\sigma_n^2}\right], \tag{2.38}$$

with spin cut-off, $\sigma_n^2 = 0.24nA^{2/3}$ [26].

A ν-step cross section can be described in a recursive way as the convolution of a single-step and a $(\nu-1)$-step cross section:

$$\frac{d^2\sigma^{(\nu)}(E,\Omega \leftarrow E_0,\Omega_0)}{d\Omega dE} = \frac{m}{4\pi^2\hbar^2}\int d\Omega_{\nu-1}\int dE_{\nu-1}E_{\nu-1}$$
$$\times\frac{d^2\sigma^{(1)}(E,\Omega \leftarrow E_{\nu-1},\Omega_{\nu-1})}{d\Omega dE}\frac{d^2\tilde{\sigma}^{(\nu-1)}(E_{\nu-1},\Omega_{\nu-1} \leftarrow E_0,\Omega_0)}{d\Omega_{\nu-1}dE_{\nu-1}} \tag{2.39}$$

The expression for $\frac{d^2\tilde{\sigma}^{(\nu-1)}(E_{\nu-1},\Omega_{\nu-1}\leftarrow E_0,\Omega_0)}{d\Omega_{\nu-1}dE_{\nu-1}}$ can easily be found by comparing equations (2.34) and (2.39).

To obtain the total MSD cross sections, individual ν-step cross sections are added together and the coupling magnitude v is determined by fitting the total MSD cross section to experimental data.

2.2 MSD Processes with Two Continuum Particles

2.2.1 Introduction

MSD theory as formulated by FKK describes processes with one and only one continuum particle at all stages of a multistep process. This is equivalent to assuming

that the amount of energy imparted to an individual nucleon in any given step is less than the energy required to free a bound nucleon.

Even though the cross section for single continuum particle MSD processes is given by the FKK, an abbreviated derivation was presented in Section (2.1) to facilitate the derivation of formalism for two or more continuum particles.

It has been recognized by many authors that multiple emission from pre-equilibrium processes becomes important at projectile energies of few tens of MeV. These processes have been described in a variety of ways, including semi-classical exciton [35] and hybrid [36] models, and the internuclear cascade model.

Furthermore, there have been several previous attempts to generalize FKK theory to describe multiple pre-equilibrium emission, most notably by Ciangaru [18] and Chadwick *et al.* [33]. Ciangaru developed an FKK extension which describes two-particle emisssion processes via an initial knock-out process. However, this formalism stops short of describing pre-equilibrium emission of three or more particles. Chadwick *et al.* used clever application of phase space arguments to derive an expression for two particle pre-equilibrium emission processes without introducing any quantities not already used by ordinary FKK MSD theory.

2.2.2 The Chaining Hypothesis Revisited

The chaining hypothesis is applied to processes with more than one particle in the continuum in the same way it was applied to processes with a single particle in the continuum. To reiterate, P subspaces differing by at most at most a single particle hole pair can be reached by a single step reaction. (This is a consequence of the assumption that only one pair of nucleons interacts in each single step reaction; see discussion in Subsection 2.1.2). Unlike before, the struck particle may receive sufficient energy to become unbound, that is, a continuum particle. Thus, in a knock-out process the projectile-target system changes its configuration from 1 continuum particle and no particle-hole pairs to 2 continuum particles and 1 nuclear hole. We introduce a superscript $P_\mu^{(\mathcal{N})}$, $\mathcal{N}$ denoting the number of continuum particles at μ^{th} step.

2.2.3 The Spectator Hypothesis

Like before, the effect of the spectator hypothesis is that a single step direct cross section is independent of the stage number, or equivalently, independent of the number of particle-hole pairs created by prior single step reactions. Obviously this approximation fails when the number of steps becomes very large ($\sqrt{A}$), but for all practical purposes this is a very good approximation because the number of steps contributing significantly is on the order of five steps for most processes.

The reason why the spectator hypothesis is applicable only to MSD processes is that the time scale on which a single step direct process occurs is too short for the transferred energy to be distributed among all existing particle-hole pairs. On the other hand, the reaction time for MSC processes is long enough for the transferred energy to equilibrate with all particle-hole pairs; consequently the spectator hypothesis is not applicable. That is why in multistep compound processes all particles and holes have to be taken into account.

Application of the spectator hypothesis to processes with more than one continuum particle is especially beneficial in simplifying the cross section. For example, for two continuum particles, a two step process in which each particle scatters only once can take place in two different ways. However, by the spectator hypothesis these two processes are identical and can be accounted for by a combinatorial factor, which in this case is 2.

To generalize this argument consider a process with two continuum particles in which one particle undergoes μ and the other one ν single step direct scatterings. Then there are $\frac{(\mu+\nu)!}{\mu!\nu!}$ different ways this process can occur, all of which are equivalent. Thus we need to evaluate only one such process and multiply it by this combinatorial factor, as opposed to evaluating as many different processes. The computational savings increases for greater number of steps. This approximation was not applied in the previous treatment of two continuum particle processes by Ciangaru [18], consequently rendering the final expression cumbersome to evaluate.

In practical terms, the application of the spectator hypothesis is equivalent to

dropping subspace indices on a single step transition amplitude (2.33) [5]. The remaining momentum indices do not matter since these momenta are integration variables.

The spectator hypothesis is a consequence of the following approximation

$$\rho_{(j+1)p(j+1)h}(U_{j+1})\ \langle|\bar{v}_{\alpha_{j+1}\alpha_j}(\mathbf{k}_{j+1},\mathbf{k}_j)|^2\rangle\ \approx\ \rho_{1p1h}(U_{j+1}-U_j)\ \langle|\bar{v}_{\alpha_1 i}(\mathbf{k}_{j+1},\mathbf{k}_j)|^2\rangle \quad (2.40)$$

which says that a product of a large density $\rho_{(j+1)p(j+1)h}(U_{j+1})$ and a small matrix element $\langle|\bar{v}_{\alpha_{j+1}\alpha_j}(\mathbf{k}_{j+1},\mathbf{k}_j)|^2\rangle$ is equal to a product of a small density $\rho_{1p1h}(U_{j+1}-U_j)$ and a large matrix element $\langle|\bar{v}_{\alpha_1 i}(\mathbf{k}_{j+1},\mathbf{k}_j)|^2\rangle$.

$$U_j = E - \frac{\hbar^2 k_j^2}{2m} \quad (2.41)$$

is the total deposited energy after j steps and $(U_{j+1}-U_j)$ is the amount of energy deposited in the $(j+1)$ step only.

2.2.4 Various Reaction Scenarios

For the sake of a clearer exposition we will start out with the simplest multi-particle final state process: a two-particle final state process. In such a process, the incoming projectile may create the second continuum particle in its first interaction with nucleus, or it may undergo a regular direct scattering before creating the second particle. However, the projectile is most likely to create the second free particle when its energy is the highest, i.e. in the first step.

Let us denote the subspace in which the creation of the second continuum particle occurs with an index ν, and let us use μ to denote the final stage of the residual nucleon after the emission has occurred. Here ν is equivalent to the number of single-step direct processes undergone by the projectile before it creates the second continuum particle. To interpret μ properly, we recognize that with two particles in the continuum either one of them can undergo a single step scattering and by doing

[5] 'Dropping' is a slight misnomer, since the amplitude at any stage is replaced with the amplitude at the first stage. Because the amplitude is thus always the same, we can omit this index.

so move the process from one subspace to a neighboring one. There are many ways of reaching subspace μ from subspace ν, the number of which can be expressed by the combinatorial factor $2^{(\mu-\nu)}$.

In a general multiparticle process, when there are several continuum particles at some arbitrary stage of a reaction, each one of the continuum particles may interact with any of the bound nucleons. In such a reaction, a continuum particle may create an additional unbound nucleon, or it may simply create a particle-hole pair via a usual single-step process. Many particle processes are treated in Section 2.3.

2.2.5 Two-particle cross section

As was the case for a single particle MSD scattering, the transition matrix element for two-particle MSD scattering is written as a sum of $\mathcal{T}_{fi}$ elements, each element corresponding to scattering over different number of steps.

$$\mathcal{T} = \sum_{\mu}^{\infty} \mathcal{T}^{\mu} \tag{2.42}$$

Wavefunctions used to expand the two-particle Green's function will now have to represent two continuum particles. We write the two particle wavefunction as a product of two single particle wavefunctions, which in the case of two indistinguishable nucleons would have to be properly antisymmetrized. However, we may proceed in the derivation without concerning ourselves with antisymmetrization of wavefunctions because effects of antisymmetrization can made invisible by assuming the v-matrix elements are evaluated with antisymmetrized wavefunctions. In other words, whether or not the wavefunctions are symmetrized affects only the v-matrix elements, not the overall structure.

$$G_{P_n^{(2)}} = \sum_{\alpha} \int\int \frac{d\mathbf{k}_n}{(2\pi)^3} \frac{d\mathbf{K}_n}{(2\pi)^3} \frac{|\Psi(\mathbf{k}_n)^{(+)}\Psi(\mathbf{K}_n)^{(+)}\rangle_{\alpha}(\tilde{\Psi}(\mathbf{k}_n)^{(+)}\tilde{\Psi}(\mathbf{K}_n)^{(+)}|_{\alpha}}{E^{(+)} - \frac{\hbar^2}{2m}(k_n^2 + K_n^2) - \mathcal{E}_{n\alpha}} \tag{2.43}$$

In the above equation $\mathbf{k}_n$ is the momentum of one of the two particles in the

continuum and $\mathbf{K}_n$ is the momentum of the other one. Subscript n denotes a $P_n^{(2)}$ subspace in which the sum over nuclear states denoted by α, is performed. If the two particles in the continuum whose momenta are denoted by $\mathbf{k}_n$ and $\mathbf{K}_n$ are indistinguishable then we cannot say which one of them is the projectile and which one is the ejected nucleon. In cases where knock-out processes are significant, however, the projectile's energy will usually be much higher than that of the knocked out nucleon, so that the particle with higher energy is most likely to be the projectile.

The next step in the derivation is identical to that given by equations (2.22) and (2.23) in derivation of single particle MSD processes. We proceed as before by inserting the expansion of the Green's function (2.43) into Equation (2.23) as follows:

$$\begin{aligned} v_{P_2^{(2)}P_1^{(2)}} G_{P_1^{(2)}} v_{P_1^{(2)}i} |\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}| v^*_{iP_1^{(2)}} G^*_{P_1^{(2)}} v^*_{P_1^{(2)}P_2^{(2)}} \quad = \\ \sum_{\alpha\alpha'} \int\int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{K}_1}{(2\pi)^3} \int\int \frac{d\mathbf{k}'_1}{(2\pi)^3}\frac{d\mathbf{K}'_1}{(2\pi)^3} \\ \times \bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2;\mathbf{k}_1,\mathbf{K}_1) \frac{1}{E^{(+)} - (\hbar/2m)(k_1^2 + K_2^1) - \mathcal{E}_{1\alpha}} \bar{v}_{\alpha i}(\mathbf{K}_1,\mathbf{k}_1;\mathbf{k}_i) \quad (2.44) \\ \times \bar{v}_{i\alpha'}(\mathbf{k}_i;\mathbf{k}'_1,\mathbf{K}'_2)^* \frac{1}{E^{(-)} - (\hbar/2m)(k_1'^2 + K_1'^2) - \mathcal{E}_{1\alpha'}} \bar{v}_{\alpha'\gamma'}(\mathbf{k}'_1,\mathbf{K}'_1;\mathbf{k}'_2,\mathbf{K}'_2)^* \end{aligned}$$

where

$$\begin{aligned} \bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2;\mathbf{k}_1,\mathbf{K}_1) &= {}_\gamma\langle\tilde{\Psi}(\mathbf{k}_2)^{(+)}\tilde{\Psi}(\mathbf{K}_2)^{(+)}|v|\Psi(\mathbf{k}_1)^{(+)}\Psi(\mathbf{K}_1)^{(+)}\rangle_\alpha \\ \bar{v}_{\alpha i}(\mathbf{K}_1,\mathbf{k}_1;\mathbf{k}_i) &= {}_\alpha\langle\tilde{\Psi}(\mathbf{k}_1)^{(+)}\tilde{\Psi}(\mathbf{K}_1)^{(+)}|v|\phi(\mathbf{k}_i)^{(+)}\rangle \end{aligned} \quad (2.45)$$

Matrix element $\bar{v}_{\alpha i}(\mathbf{K}_1,\mathbf{k}_1;\mathbf{k}_i)$ contains information about the bound nucleon which is being knocked out, since this matrix element is implicitly averaged over momenta of the bound nucleon.

By the random phase hypothesis the only non-vanishing terms in the summation over α and α' are those for which $\alpha = \alpha'$. Therefore we replace α' with α and drop

the summation over α'.

The remaining sum over α can be cast into an energy integral $\sum_\alpha = \int d\mathcal{E}\rho_{1h}(\mathcal{E})$ where $\rho_{1h}(\mathcal{E})$ is density of one-hole states at energy $\mathcal{E}$. The conversion to an energy integral has to be done with an understanding that the quantities inside the sum will be averaged over all states at a particular energy, i.e. averaged over all states that are counted in the level density at a particular energy. Integration over energy yields:

$$\begin{aligned} v_{P_2^{(2)}P_1^{(2)}}G_{P_1^{(2)}}v_{P_1^{(2)}i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v^*_{iP_1^{(2)}}G^*_{P_1^{(2)}}v^*_{P_1^{(2)}P_2^{(2)}} \quad = \\ \int\int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{K}_1}{(2\pi)^3}\int\int \frac{d\mathbf{k}'_1}{(2\pi)^3}\frac{d\mathbf{K}'_1}{(2\pi)^3} \\ \int d\mathcal{E}_{1\alpha}\rho_{1h}(\mathcal{E}_{1\alpha})\frac{\mathcal{M}(\mathcal{E}_{1\alpha},\mathbf{k}_1,\mathbf{K}_1,\mathbf{k}'_1,\mathbf{K}'_1)}{[E^{(+)}-(\hbar/2m)(k_1^2+K_1^2)-\mathcal{E}_{1\alpha}][E^{(-)}-(\hbar/2m)(k_1'^2+K_1'^2)-\mathcal{E}_{1\alpha}]} \end{aligned} \tag{2.46}$$

where for the sake of compactness we have used the following definition:

$$\begin{aligned} \mathcal{M}(\mathcal{E}_{1\alpha},\mathbf{k}_1,\mathbf{K}_1,\mathbf{k}'_1,\mathbf{K}'_1) \;=\; & \bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2;\mathbf{k}_1,\mathbf{K}_1)\bar{v}_{\alpha i}(\mathbf{K}_1,\mathbf{k}_1;\mathbf{k}_i) \\ & \bar{v}_{i\alpha}(\mathbf{k}_i;\mathbf{k}'_1,\mathbf{K}'_2)^*\bar{v}_{\alpha\gamma}(\mathbf{k}'_1,\mathbf{K}'_1;\mathbf{k}'_2,\mathbf{K}'_2)^* \end{aligned} \tag{2.47}$$

First we evaluate the integral over $\mathcal{E}_{1\alpha}$ in (2.46), which we denote by I:

$$I \equiv \int d\mathcal{E}_{1\alpha}\rho_{1h}(\mathcal{E}_{1\alpha})\frac{\mathcal{M}(\mathcal{E}_{1\alpha},\mathbf{k}_1,\mathbf{K}_1,\mathbf{k}'_1,\mathbf{K}'_1)}{[E^{(+)}-(\hbar/2m)(k_1^2+K_1^2)-\mathcal{E}_{1\alpha}][E^{(-)}-(\hbar/2m)(k_1'^2+K_1'^2)-\mathcal{E}_{1\alpha}]} \tag{2.48}$$

We evaluate the above integral in the complex plane by the usual residue method:

$$I \;=\; \rho_{1h}(U_1)\,\mathcal{M}(U_1,\mathbf{k}_1,\mathbf{K}_1,\mathbf{k}'_1,\mathbf{K}'_1)\,\frac{2\pi i}{(\hbar^2/2m)((k_1^2+K_1^2)-(k_1'^2+K_1'^2))} \tag{2.49}$$

where

$$U_1 = E - \frac{\hbar^2}{2m}(k_1^2 + K_1^2) - E_b \tag{2.50}$$

is the energy transferred to the nucleus in this step, and E_b is the nuclear binding energy. Next, we apply the same arguments employed in the single particle MSD derivation, equations (2.26) - (2.27), in order to argue that the the above integral can be approximated by its δ-function contribution:

$$I = 2\pi^2 \rho_{1h}(U_1)\, \mathcal{M}(U_1, \mathbf{k}_1, \mathbf{K}_1, \mathbf{k}_1', \mathbf{K}_1')\, \delta\left(\frac{\hbar^2}{2m}[(k_1^2 + K_1^2) - (k_1'^2 + K_1'^2)]\right) \tag{2.51}$$

Substituting (2.49) into (2.46) and rewriting $\mathcal{M}$ in terms of v's we obtain

$$\begin{aligned} v_{P_2^{(2)}P_1^{(2)}} G_{P_1^{(2)}} v_{P_1^{(2)}i} |\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}| v^*_{iP_1^{(2)}} G^*_{P_1^{(2)}} v^*_{P_1^{(2)}P_2^{(2)}} \;=\; & \int\int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{K}_1}{(2\pi)^3}\int\int \frac{d\mathbf{k}_1'}{(2\pi)^3}\frac{d\mathbf{K}_1'}{(2\pi)^3} \\ & \times\langle \bar{v}_{\gamma\alpha}(\mathbf{k}_2, \mathbf{K}_2; \mathbf{k}_1, \mathbf{K}_1)\bar{v}_{\alpha i}(\mathbf{K}_1, \mathbf{k}_1; \mathbf{k}_i)\bar{v}^*_{i\alpha}(\mathbf{k}_i; \mathbf{k}_1', \mathbf{K}_2')\bar{v}^*_{\alpha\gamma}(\mathbf{k}_1', \mathbf{K}_1'; \mathbf{k}_2', \mathbf{K}_2')\rangle \\ & \times\ \rho_{1h}(U_1)\ 2\pi^2\ \delta\left(\frac{\hbar^2}{2m}[(k_1^2 + K_1^2) - (k_1'^2 + K_1'^2)]\right) \end{aligned} \tag{2.52}$$

Evaluating integrals over primed momenta in the above expression has to be done with care. Because of the random phase approximation the argument of the delta function in the above equation vanishes in two possible ways: 1) $k_1^2 = k_1'$ and $K_1^2 = K_1'^2$, or 2) $k_1^2 = K_1'^2$ and $K_1^2 = k_1'^2$. If the two particle wave-functions are properly symmetrized, each of the two ways will make an identical contribution, resulting in an extra factor of two.

To avoid pitfalls leading to the spurious factor of two one converts momentum integration from the individual momenta $\mathbf{k}_1$ and $\mathbf{K}_1$, to the relative and the total momentum ($\mathbf{q}_1$ and $\mathbf{Q}_1$, respectively) defined as:

$$\mathbf{q}_1 \;\equiv\; \frac{(m\mathbf{k}_1 - M\mathbf{K}_1)}{m+M} \overset{m=M}{\rightarrow} \frac{(\mathbf{k}_1 - \mathbf{K}_1)}{2} \tag{2.53}$$

$$\mathbf{Q}_1 \;\equiv\; \mathbf{k}_1 + \mathbf{K}_1 \tag{2.54}$$

Using the above notation and the definition of the reduced mass $\mu = mM/(m+M) \stackrel{m=M}{\rightarrow} m/2$ the delta function can be written as

$$\delta\left(\frac{\hbar^2}{2m}[(k_1^2+K_1^2)-(k_1'^2+K_1'^2)]\right) = \delta\left(\frac{\hbar^2}{2\mu}(q_1^2-q_1'^2)+\frac{\hbar^2}{2(m+M)}(Q_1^2-Q_1'^2)\right) \tag{2.55}$$

When the δ-function is written in terms of q and Q it becomes more transparent that its argument vanishes only when $q = q'$ and $Q = Q'$. It makes no physical sense to equate q and Q since they are two different physical quantities. Therefore, the delta function will give a non-zero contribution only when $q_1^2 = q_1'^2$ and $Q_1^2 = Q_1'^2$, and consequently there is no factor of two.

Furthermore, because of the random phase hypothesis the above integrand gives a non-zero contribution only when $\mathbf{q}_1 = \mathbf{q}_1'$ and $\mathbf{Q}_1 = \mathbf{Q}_1'$. When this is the case, however, the argument of the delta function above will be zero giving a non-zero contribution to the integral.

So, when properly evaluated, the integral in (2.52) becomes:

$$\begin{aligned} v_{P_2^{(2)}P_1^{(2)}} G_{P_1^{(2)}} v_{P_1^{(2)}i} |\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}| v^*_{iP_1^{(2)}} G^*_{P_1^{(2)}} v^*_{P_1^{(2)}P_2^{(2)}} \;=\; & \int\int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{K}_1}{(2\pi)^3} \\ & \times\langle \bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2;\mathbf{k}_1,\mathbf{K}_1)|\bar{v}_{\alpha i}(\mathbf{K}_1,\mathbf{k}_1;\mathbf{k}_i)|^2 \bar{v}^*_{\alpha\gamma}(\mathbf{k}_1,\mathbf{K}_1;\mathbf{k}_2',\mathbf{K}_2')\rangle \\ & \times 2\pi^2\, \rho_{2p}(k_1,K_1)\; \rho_{1h}(U_1) \end{aligned} \tag{2.56}$$

The above equation is most easily understood when compared with its single continuum particle counterpart given in Equation (2.30). Instead of only one momentum integration in (2.30), there is one momentum integration for each continuum particle. The density of states is modified to reflect the fact that after the first step there are two continuum particles and a 1h nuclear state, as opposed to a single continuum particle and a 1p1h nuclear state in (2.30).

We proceed by evaluating the next step in the multistep reaction. If we presume that no additional continuum particles will be created in the multi-step process under consideration, the next step entails multiplying the above equation $v_{P_3^{(2)}P_2^{(2)}}G_{P_2^{(2)}}$ to the left and $G^*_{P_2^{(2)}}v^*_{P_2^{(2)}P_3^{(2)}}$ to the right. On the other hand if an additional particle were to be created in the second step, then we would multiply the above equation by $v_{P_3^{(3)}P_2^{(3)}}G_{P_2^{(3)}}$ to the left and $G^*_{P_2^{(3)}}v^*_{P_2^{(3)}P_3^{(3)}}$ to the right. The cross sections coming from these two different possibilities have to be incoherently added at the end. For the sake of exposition, we proceed, evaluating the case in which no additional particles are created, i.e. the two continuum particles undergo simple MSD scattering over the nucleus.

$$
\begin{aligned}
v_{P_3^{(2)}P_2^{(2)}}G_{P_2^{(2)}}v_{P_2^{(2)}P_1^{(2)}}G_{P_1^{(2)}}v_{P_1^{(2)}i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v^*_{iP_1^{(2)}}G^*_{P_1^{(2)}}v^*_{P_1^{(2)}P_2^{(2)}}G^*_{P_2^{(2)}}v^*_{P_2^{(2)}P_3^{(2)}} \quad = \\
\sum_{\gamma,\gamma'}\int\int\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\mathbf{K}_2}{(2\pi)^3}\int\int\frac{d\mathbf{k}'_2}{(2\pi)^3}\frac{d\mathbf{K}'_2}{(2\pi)^3}\int\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{K}_1}{(2\pi)^3} \\
\times\rho_{1h}(U_1)\ \rho_{2p}(E_{k_1},E_{K_1})\langle|\bar{v}_{\alpha i}(\mathbf{K}_1,\mathbf{k}_1;\mathbf{k}_i)|^2\rangle \qquad (2.57)\\
\times\bar{v}_{\lambda\gamma}(\mathbf{k}_3,\mathbf{K}_3;\mathbf{k}_2,\mathbf{K}_2)\frac{1}{E-(\hbar/2m)(k_2^2+K_2^2)-\mathcal{E}_{2\gamma}}\bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2;\mathbf{k}_1,\mathbf{K}_1) \\
\times\bar{v}^*_{\alpha\gamma'}(\mathbf{k}_1,\mathbf{K}_1;\mathbf{k}'_2,\mathbf{K}'_2)\frac{1}{E-(\hbar/2m)(k_2'^2+K_2'^2)-\mathcal{E}_{2\gamma'}}\bar{v}^*_{\gamma'\lambda'}(\mathbf{k}'_2,\mathbf{K}'_2;\mathbf{k}'_3,\mathbf{K}'_3)
\end{aligned}
$$

Furthermore, we do not consider processes in which the continuum particles interact with each other since the probability of this is much smaller than the probability of interacting with a bound nucleon. This approximation will hold if the number of continuum particles is much smaller than the atomic number of the target nucleus. The consequence of this approximation is that the matrix elements $\bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2;\mathbf{k}_1,\mathbf{K}_1)$ can be written as a sum of the matrix elements corresponding to interactions between each continuum particle and a bound nucleon.

$$\bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2;\mathbf{k}_1,\mathbf{K}_1) = (2\pi)^3[\bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{K}_2)\,\delta(\mathbf{k}_1-\mathbf{K}_1)+\bar{v}_{\gamma\alpha}(\mathbf{k}_1,\mathbf{K}_1)\,\delta(\mathbf{k}_2-\mathbf{K}_2)]$$
$$\bar{v}^*_{\alpha\gamma}(\mathbf{k}_1,\mathbf{K}_1;\mathbf{k}'_2,\mathbf{K}'_2) = (2\pi)^3[\bar{v}^*_{\gamma\alpha}(\mathbf{k}'_2,\mathbf{K}'_2)\,\delta(\mathbf{k}_1-\mathbf{K}_1)+\bar{v}^*_{\gamma\alpha}(\mathbf{k}_1,\mathbf{K}_1)\,\delta(\mathbf{k}'_2-\mathbf{K}'_2)] \tag{2.58}$$

Inserting the above expressions for the v-matrix elements and expanding the Green's functions in equation (2.58), followed by a series of random phase approximations that closely parallel the derivation of the cross section for single particle MSD processes [6] we easily obtain:

$$\begin{aligned} v_{P_3^{(2)}P_2^{(2)}}G_{P_2^{(2)}}v_{P_2^{(2)}P_1^{(2)}}G_{P_1^{(2)}}v_{P_1^{(2)}i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v^*_{iP_1^{(2)}}G^*_{P_1^{(2)}}v^*_{P_1^{(2)}P_2^{(2)}}G^*_{P_2^{(2)}}v^*_{P_2^{(2)}P_3^{(2)}} = \\ \int\int\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\mathbf{K}_2}{(2\pi)^3}\int\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{K}_1}{(2\pi)^3} \\ \times 2\pi^2\,\rho_{1h}(U_1)\;\rho_{2p}(E_{k_1},E_{K_1})\langle|\bar{v}_{\alpha i}(\mathbf{K}_1,\mathbf{k}_1;\mathbf{k}_i)|^2\rangle \\ \times[\;\;2\pi^2\rho_{1p2h}(U_2)\;\rho(E_{k_2})\langle|\bar{v}_{\gamma\alpha}(\mathbf{k}_2,\mathbf{k}_1)|^2\rangle\,\delta(\mathbf{K}_2-\mathbf{K}_1) \\ +2\pi^2\rho_{1p2h}(U_2)\;\rho(E_{K_2})\langle|\bar{v}_{\gamma\alpha}(\mathbf{K}_2,\mathbf{K}_1)|^2\rangle\,\delta(\mathbf{k}_2-\mathbf{k}_1)] \\ \times\bar{v}_{\lambda\gamma}(\mathbf{k}_3,\mathbf{K}_3;\mathbf{k}_2,\mathbf{K}_2)\bar{v}^*_{\gamma'\lambda'}(\mathbf{k}'_2,\mathbf{K}'_2;\mathbf{k}'_3,\mathbf{K}'_3) \end{aligned} \tag{2.59}$$

where

$$U_2 = E-\frac{\hbar^2}{2m}(k_2^2+K_2^2)-E_b \tag{2.60}$$

is the total energy deposited to the nucleus after the first two steps. This energy is distributed among 1 particle and 2 holes, as is indicated by the subscript in $\rho_{1p2h}(U_2)$.

[6] The only application of RPA that was not already encountered in single particle MSD derivation is toward elimination of cross terms which occur when the two v matrix elements defined in Equation (2.58) are multiplied.

Subsequent steps are evaluated by repeating the above process, i.e. Eq. (2.59) is multiplied by $v_{P_4^{(2)}P_3^{(2)}}G_{P_3^{(2)}}$ from the left and $G^*_{P_3^{(2)}}v^*_{P_3^{(2)}P_4^{(2)}}$ to the right and so forth. Contributions from various steps are added together and the resulting $|\mathcal{T}_{fi}^{MSD}|^2$ is then used to obtain the two-particle MSD cross section:

$$\frac{d^4\sigma^{MSD}(\mathbf{k}_f,\mathbf{K}_f\leftarrow\mathbf{k}_i)}{dE_{k_f}d\Omega_{k_f}dE_{K_f}d\Omega_{K_f}}=\frac{2\pi}{\hbar}\,\frac{m}{\hbar k_i}\,\rho(k_f)\,\rho(K_f)\,|\mathcal{T}_{fi}^{MSD}|^2 \tag{2.61}$$

yielding

$$\begin{aligned}\frac{d^4\sigma^{MSD}(\mathbf{k}_f,\mathbf{K}_f\leftarrow\mathbf{k}_i)}{dE_{k_f}d\Omega_{k_f}dE_{K_f}d\Omega_{K_f}} &= \sum_n\int\int\frac{d\mathbf{k}_n}{(2\pi)^3}\frac{d\mathbf{K}_n}{(2\pi)^3}\cdots\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{K}_1}{(2\pi)^3}\\ &\times\frac{d^4w^{(1)}(\mathbf{k}_f,\mathbf{K}_f\leftarrow\mathbf{k}_n,\mathbf{K}_n)}{dE_{k_f}d\Omega_{k_f}dE_{K_f}d\Omega_{K_f}}\cdots\frac{d^4\tilde{w}^{(1)}(\mathbf{k}_2,\mathbf{K}_2\leftarrow\mathbf{k}_1,\mathbf{K}_1)}{dE_{k_2}d\Omega_{k_2}dE_{K_2}d\Omega_{K_2}}\\ &\times\frac{d^4\tilde{\sigma}^{(1)}(\mathbf{k}_1,\mathbf{K}_1\leftarrow\mathbf{k}_i)}{dE_{k_1}d\Omega_{k_1}dE_{K_1}d\Omega_{K_1}}\end{aligned} \tag{2.62}$$

where

$$\begin{aligned}\frac{d^4w^{(1)}(\mathbf{k}_{j+1},\mathbf{K}_{j+1}\leftarrow\mathbf{k}_j,\mathbf{K}_j)}{dE_{k_{j+1}}d\Omega_{k_{j+1}}dE_{K_{j+1}}d\Omega_{K_{j+1}}} &= \Bigg[\frac{d^2w^{(1)}(\mathbf{k}_{j+1}\leftarrow\mathbf{k}_j)}{dE_{k_{j+1}}d\Omega_{k_{j+1}}}\delta(\mathbf{K}_{j+1}-\mathbf{K}_j)\\ &+\frac{d^2w^{(1)}(\mathbf{K}_{j+1}\leftarrow\mathbf{K}_j)}{dE_{K_{j+1}}d\Omega_{K_{j+1}}}\delta(\mathbf{k}_{j+1}-\mathbf{k}_j)\Bigg]\end{aligned} \tag{2.63}$$

and

$$\frac{d^2w^{(1)}(\mathbf{k}_{j+1}\leftarrow\mathbf{k}_j)}{dE_{k_{j+1}}d\Omega_{k_{j+1}}}=2\pi^2\rho_{(j+1)p(j+1)h}(U_{j+1})\;\rho_{1p}(E_{k_{j+1}})\langle|\bar{v}_{\alpha_{j+1}\alpha_j}(\mathbf{k}_{j+1},\mathbf{k}_j)|^2\rangle \tag{2.64}$$

is a single particle single step transition amplitude. $E_{k_{j+1}}=\frac{\hbar^2}{2m}k^2_{j+1}$ is the energy of the continuum particles after $j+1$ scatterings, and the knock-out process cross section is:

$$\frac{d^4\tilde{\sigma}^{(1)}(\mathbf{k}_1, \mathbf{K}_1 \leftarrow \mathbf{k}_i)}{dE_{k_1} d\Omega_{k_1} dE_{K_1} d\Omega_{K_1}} \equiv \frac{2\pi m_0}{\hbar^2 k_i} \rho_{1h}(U_1)\, \rho_{2p}(E_{k_{j+1}}, E_{K_{j+1}})\, \langle |\bar{v}_{\alpha_1 i}(\mathbf{K}_1, \mathbf{k}_1; \mathbf{k}_i)|^2 \rangle \quad (2.65)$$

" ~ " over σ indicates that proper boundary conditions are assumed in the matrix elements of v. For discussion on proper matrix elements see the discussion following Equation (2.18).

The expression for the cross section can be simplified if the transition amplitude for scattering at a stage j is approximated by the transition amplitude for scattering at the first stage. This is equivalent to stating that the particles and holes existing at any given stage j act as spectators, and consequently the continuum particle(s) do not interact with them. This approximation breaks down for stages where the number of particle-hole pairs is much greater than one or comparable to the atomic number of a target nucleus. Fortunately, it has been shown by many authors than the processes with increasing number of steps contribute a rapidly diminishing fraction of the total cross section for heavy nuclei. The spectator hypothesis was discussed Subsection 2.2.3.

In the equation below, μ denotes the total number of steps undergone by one particle and ν by the other, so that $\mu + \nu = n$ is the total number of steps after the knock-out step:

$$\begin{aligned} \frac{d^4\sigma^{MSD}(\mathbf{k}_f, \mathbf{K}_f \leftarrow \mathbf{k}_i)}{dE_{k_f} d\Omega_{k_f} dE_{K_f} d\Omega_{K_f}} \quad = \quad & \sum_{\mu=0}^{M} \sum_{\nu=0}^{M} \frac{(\mu+\nu)!}{\mu!\ \nu!} \int\int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{d\mathbf{K}_1}{(2\pi)^3} \\ & \times \frac{d^2 w^{(\mu)}(\mathbf{k}_f \leftarrow \mathbf{k}_1)}{dE_{k_f} d\Omega_{k_f}} \ \frac{d^2 w^{(\nu)}(\mathbf{K}_f \leftarrow \mathbf{K}_1)}{dE_{K_f} d\Omega_{K_f}} \\ & \times \frac{d^4\tilde{\sigma}^{(1)}(\mathbf{k}_1, \mathbf{K}_1 \leftarrow \mathbf{k}_i)}{dE_{k_1} d\Omega_{k_1} dE_{K_1} d\Omega_{K_1}} \end{aligned} \quad (2.66)$$

where

$$\frac{d^2w^{(\mu)}(\mathbf{k}_f \leftarrow \mathbf{k}_1)}{dE_{k_f}d\Omega_{k_f}} = \int \frac{d\mathbf{k}_2}{(2\pi)^3}\cdots\frac{d\mathbf{k}_\mu}{(2\pi)^3}\left[\frac{d^2w^{(1)}(\mathbf{k}_\mu \leftarrow \mathbf{k}_{\mu-1})}{dU_\mu d\Omega_\mu}\right] \times\left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_{\mu-1} \leftarrow \mathbf{k}_{\mu-2})}{dU_{\mu-1}d\Omega_{\mu-1}}\right]\cdots\left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_2 \leftarrow \mathbf{k}_1)}{dU_2 d\Omega_2}\right] \quad (2.67)$$

is the μ-step transition matrix element. The expression for $\frac{d^2w^{(\nu)}(\mathbf{K}_f \leftarrow \mathbf{K}_1)}{dE_{K_f}d\Omega_{K_f}}$ is identical to the above if one replaces $\mathbf{k}$ by $\mathbf{K}$ and μ by ν.

Finally we express the above result in terms of cross sections:

$$\frac{d^4\sigma^{MSD}(\mathbf{k}_f, \mathbf{K}_f \leftarrow \mathbf{k}_i)}{dE_{k_f}d\Omega_{k_f}dE_{K_f}d\Omega_{K_f}} = \left(\frac{m}{4\pi\hbar^2}\right)^2 \sum_{\mu=0}^{M}\sum_{\nu=0}^{M}\frac{(\mu+\nu)!}{\mu!\ \nu!}\int dE_{K_1}E_{K_1}d\Omega_{K_1}\int dE_{k_1}E_{k_1}d\Omega_{k_1} \times\frac{d^2\sigma^{(\mu)}(\mathbf{k}_f \leftarrow \mathbf{k}_1)}{dE_{k_f}d\Omega_{k_f}}\ \frac{d^2\sigma^{(\nu)}(\mathbf{K}_f \leftarrow \mathbf{K}_1)}{dE_{K_f}d\Omega_{K_f}} \times\frac{d^4\tilde{\sigma}^{(1)}(\mathbf{k}_1, \mathbf{K}_1 \leftarrow \mathbf{k}_i)}{dE_{k_1}d\Omega_{k_1}dE_{K_1}d\Omega_{K_1}} \quad (2.68)$$

For a detailed discussion of how to account for the change from transition probability to cross section, as well as the change of integration variables from momenta to energy and angle, see Equation (B.3) and the paragraph following it.

Besides the class of processes in which the second continuum particle is created in the first step (like the one we have just developed formalism for), there is another class of processes in which the second continuum particle is created after the projectile has undergone one or more steps. In order to follow the projectile as it undergoes conventional (i.e. a single continuum particle) multi-step scattering we simply use the results from the original FKK theory. At the step at which the second continuum particle is created we have to employ the formalism for the creation of the second particle. In doing so, we must recognize that in the general expression for the the two-particle cross section, the incident momentum of the projectile is given by the momentum of the projectile after it has undergone the appropriate number of direct steps. Then, the single particle cross section (2.34) for the processes preceding creation of the second continuum particle is convoluted with the two-particle cross section (2.68).

In such a pre-scattering scenario the projectile can undergo an arbitrary number of direct steps denoted by λ. As was the case in the original FKK theory, in order to obtain the amplitude for such a process, one simply convolutes λ single step amplitudes. After the projectile undergoes λ conventional steps it has a momentum $\mathbf{k}_\lambda$ and in the next $(\lambda+1)$ step it creates the second continuum particle.

However, since the λ particle-hole pairs created by the projectile are treated as pure spectators, the two particle cross section (2.68) is a function of $\mathbf{k}_\lambda$ only and is independent of the number of steps it took to create this momentum.

The momentum at which the projectile undergoes a knock-out process, i.e. $\mathbf{k}_\lambda$, can be reached by any number of steps . Contributions coming from $\lambda = 1, 2, 3, ...$ step processes are added together, giving the FKK total MSD amplitude for a particle with final momentum $\mathbf{k}_\lambda$. We employ this observation to write the two-particle cross section with MSD prescattering as follows.

$$\frac{d^4\sigma^{MSD}(\mathbf{k}_f,\mathbf{K}_f \leftarrow \mathbf{k}_i)}{dE_{k_f}d\Omega_{k_f}dE_{K_f}d\Omega_{K_f}} = \left(\frac{m}{4\pi\hbar^2}\right)^2 \sum_{\mu=0}^{M}\sum_{\nu=0}^{M}\frac{(\mu+\nu)!}{\mu!\ \nu!}\int dE_{k'}E_{k'}d\Omega_{k'}\int dE_{K'}E_{K'}d\Omega_{K'}$$
$$\times \frac{d^2\sigma^{(\mu)}(\mathbf{k}_f \leftarrow \mathbf{k}')}{dE_{k_f}d\Omega_{k_f}}\ \frac{d^2\sigma^{(\nu)}(\mathbf{K}_f \leftarrow \mathbf{K}')}{dE_{K_f}d\Omega_{K_f}} \tag{2.69}$$
$$\times \left(\frac{m}{4\pi\hbar^2}\right)\int dE_k\ E_k d\Omega_k \frac{d^4\tilde{\sigma}^{(1)}(\mathbf{k}',\mathbf{K}' \leftarrow \mathbf{k})}{dE_{k'}d\Omega_{k'}dE_{K'}d\Omega_{K'}}\frac{d^2\tilde{\sigma}^{MSD}(\mathbf{k} \leftarrow \mathbf{k}_i)}{dE_k d\Omega_k}$$

We note that the sum which gave us the total MSD cross section in the above equation was possible because there is only one continuum particle during the pre-scattering. This is related to the fact that we cannot carry out such a summation when there is more than one continuum particle, since then we have to consider all possible orders in which the continuum particles can interact with the target.

Finally, to account for all possible scenarios leading to a two particle final state we add Equations (2.69) and (2.68) to obtain

$$\frac{d^4\sigma^{MSD}(\mathbf{k}_f,\mathbf{K}_f \leftarrow \mathbf{k}_i)}{dE_{k_f}d\Omega_{k_f}dE_{K_f}d\Omega_{K_f}} = \left(\frac{m}{4\pi\hbar^2}\right)^2 \sum_{\mu=0}^{M}\sum_{\nu=0}^{M}\frac{(\mu+\nu)!}{\mu!\ \nu!}\int dE_{k'}E_{k'}d\Omega_{k'}\int dE_{K'}E_{K'}d\Omega_{K'}$$
$$\times \frac{d^2\sigma^{(\mu)}(\mathbf{k}_f \leftarrow \mathbf{k}')}{dE_{k_f}d\Omega_{k_f}}\ \frac{d^2\sigma^{(\nu)}(\mathbf{K}_f \leftarrow \mathbf{K}')}{dE_{K_f}d\Omega_{K_f}} \tag{2.70}$$
$$\times \left[\frac{d^4\tilde{\sigma}^{(1)}(\mathbf{k}'',\mathbf{k}' \leftarrow \mathbf{k}_i)}{dE_{k'}d\Omega_{k'}dE_{K'}d\Omega_{K'}} + \left(\frac{m}{4\pi\hbar^2}\right)\int dE_k\ E_k d\Omega_k \frac{d^4\tilde{\sigma}^{(1)}(\mathbf{k}',\mathbf{K}' \leftarrow \mathbf{k})}{dE_{k'}d\Omega_{k'}dE_{K'}d\Omega_{K'}}\frac{d^2\tilde{\sigma}^{MSD}(\mathbf{k} \leftarrow \mathbf{k}_i)}{dE_k d\Omega_k}\right]$$

Unlike Ciangaru [18], in deriving the above formalism we did not assume that the knock-out collision is quasi-elastic, since the projectile energies under consideration here ($E < 100$ MeV) are too low for this assumption to be valid. In other words, we did not require the momentum of the projectile and the knocked-out nucleon to be conserved.

In our case the density of states of the continuum particles is a density of two-particle states $\rho(E_k, E_K)$ with each particle having its energy fixed; one at E_k and the other one at E_K. This two particle density of states can be expressed in terms of the one-particle densities of states if the energy bin $(\triangle E)$ is sufficiently small that the single particle density $\rho(E) \sim \sqrt{E}$ does not vary appreciable over it. Then the number of two particle states in an energy interval $\rho(E_k, E_K)\triangle E$, turns out to be $\frac{1}{2}\rho(E_k)\rho(E_K)(\triangle E)^2$ as is shown in Appendix C.

2.3 MSD Processes with more than Two Continuum Particles

In this section we will derive an expression for the cross section for processes with three or more continuum particles in the final state. The main building block of our formalism is the knock-out process via which all secondary continuum particles (i.e. all but the projectile) are created.

As we know from the previous section, the first knock-out process has to be initiated by the projectile, since it is the only continuum particle at the beginning of any process. In a knock-out process, the projectile imparts sufficient momentum to a bound nucleon to make it an unbound, continuum particle. Immediately after the first knock-out process is completed, there are two continuum particles: the projectile and the nucleon just knocked out. Either one of them may undergo additional MSD scattering over the nucleon (as was described in the previous section), or, if they have sufficiently high energy, create the third continuum particle via another knock-out process. Consequently, *any* of the three continuum particles may, in principle, knock out additional nucleons. However, since the projectile is likely to remain the most energetic particle throughout the multi-step reaction, it is the most likely candidate for creating additional continuum particles.

In order to describe regular MSD scattering that can occur between knockout processes we use a form of the interaction potential that is completely analogous to that used for two particles in Equation (2.58), i.e.

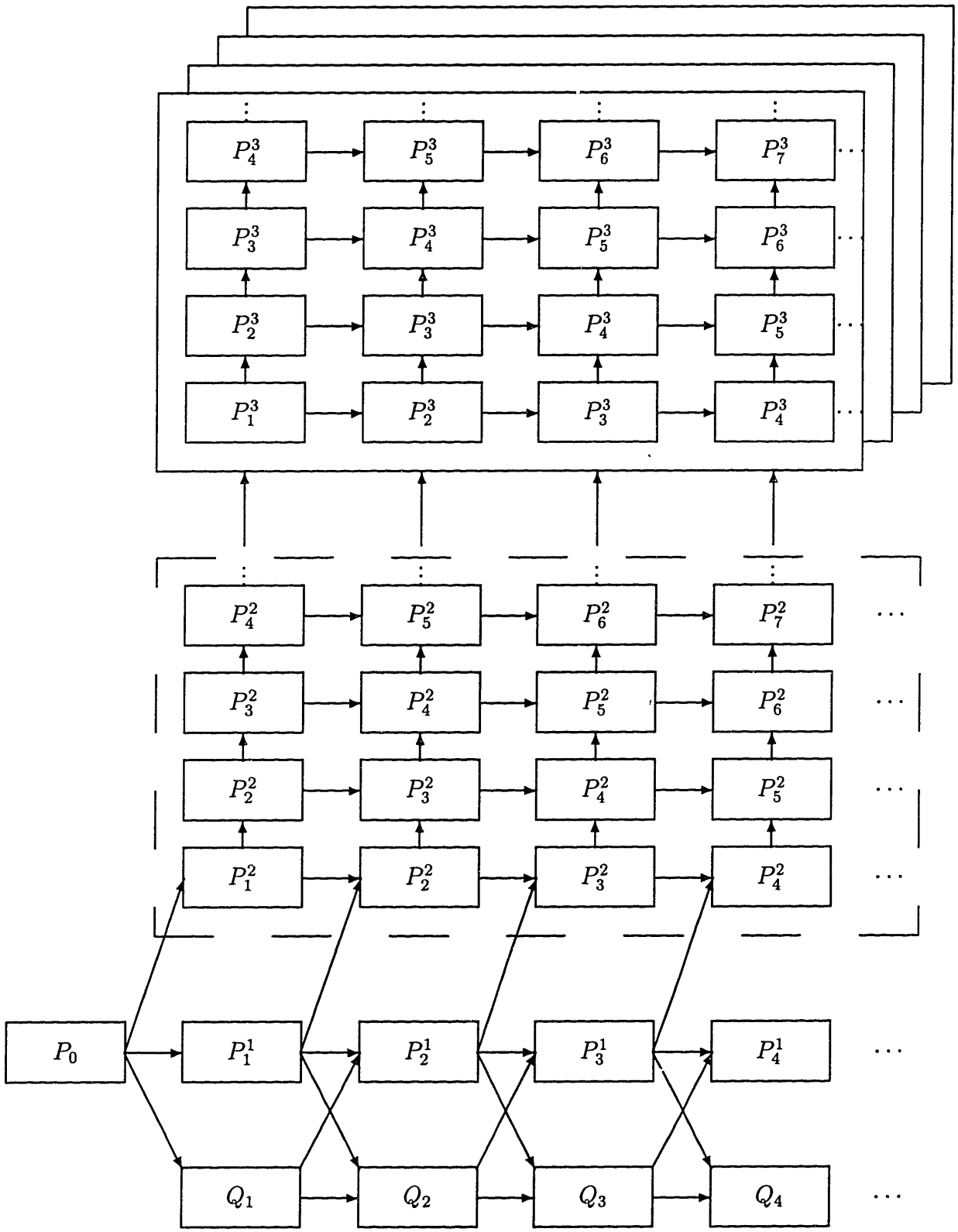

Figure 2-2: Many particle P sub-spaces are shown. Superscript ($\mathcal{N}$) refers to the number of continuum particles whereas subscript (μ) refers to the number of steps it took to reach the subspace. They are related via the number of particles and holes in the residual nucleus $h = \mu$ and $p = \mu + 1 - \mathcal{N}$. Only one 'dip' into the Q-chain is allowed.

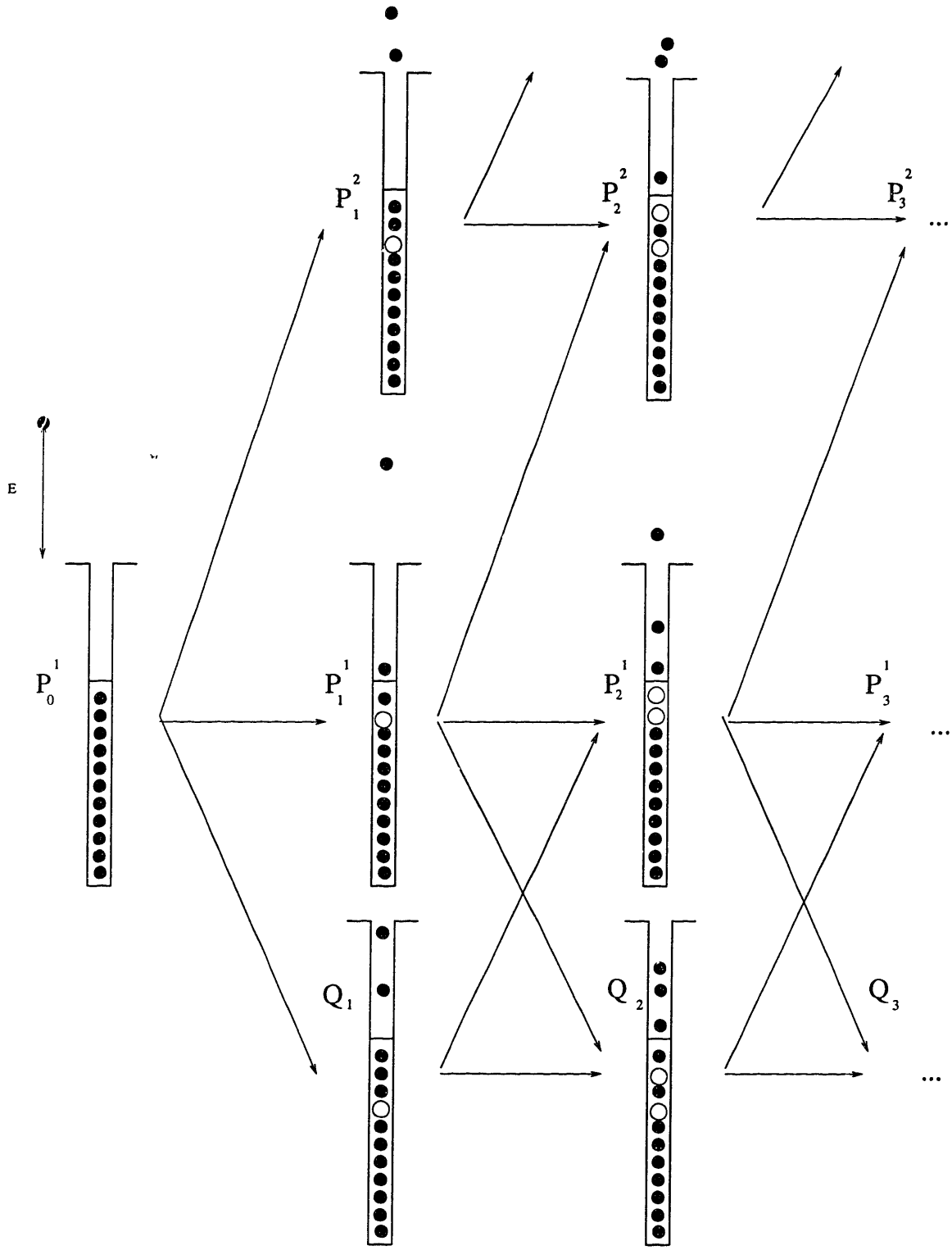

Figure 2-3: This figure shows a physical interpretation of Fig. 2-2.

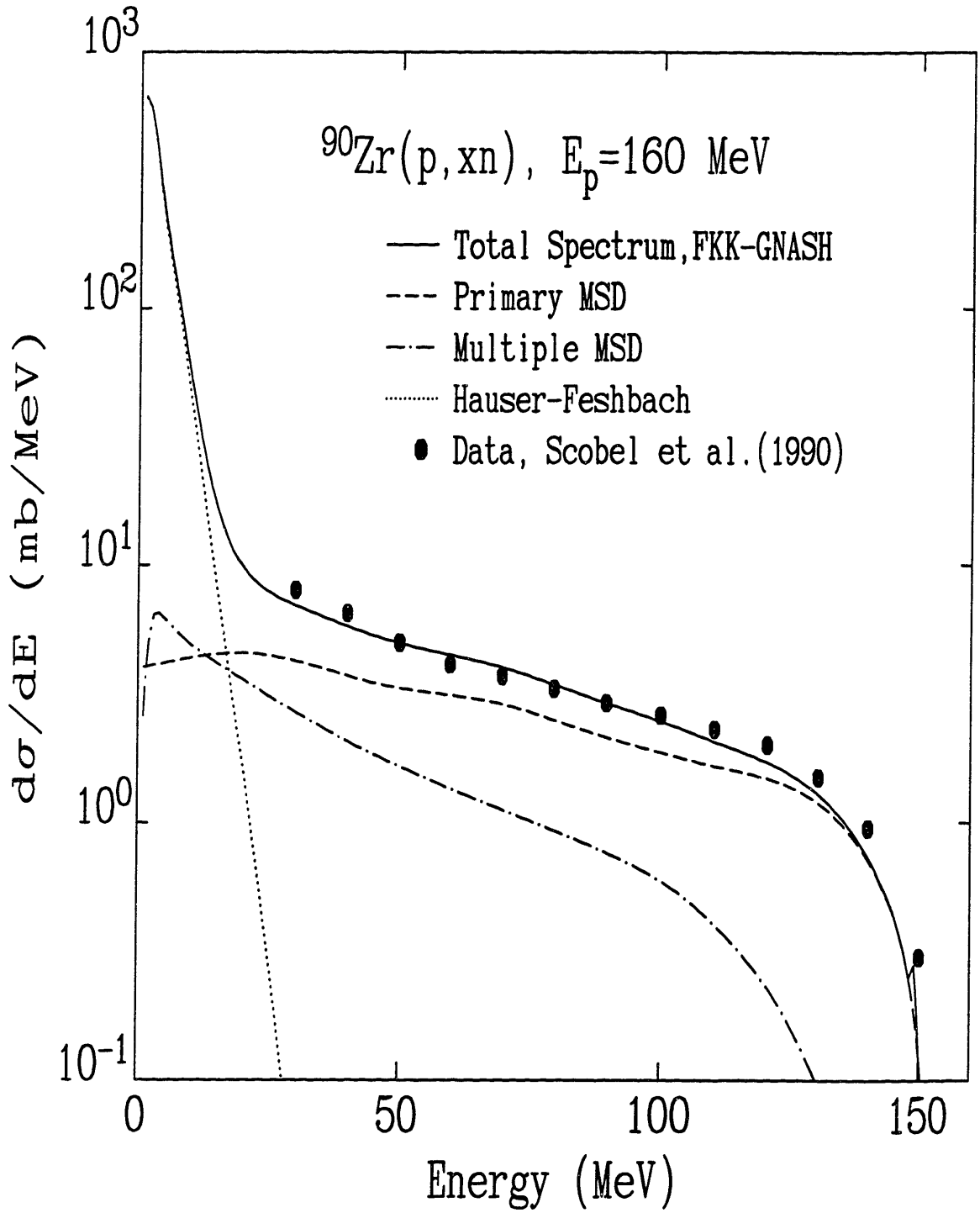

Figure 2-4: This figure shows primary (i.e. single-particle) MSD contribution to the total MSD. The difference between the total MSD emission and the primary MSD emission is accounted for by the multi-particle (i.e. secondary) emission.

$$\bar{v}_{\gamma\alpha}(\mathbf{k}_\mu^{(1)}\cdots\mathbf{k}_\mu^{(n)};\mathbf{k}_{\mu-1}^{(1)}\cdots\mathbf{k}_{\mu-1}^{(n)}) = (2\pi)^{n-1}\sum_{i=1}^{n}\left[\bar{v}_{\gamma\alpha}(\mathbf{k}_\mu^{(i)},\mathbf{k}_{\mu-1}^{(i)})\prod_{j\neq i}^{n}\delta(\mathbf{k}_\mu^{(j)}-\mathbf{k}_{\mu-1}^{(j)})\right] \tag{2.71}$$

which simply says that for single-step direct processes only one of the n continuum particles interacts with the nuclear target and the remaining $(n-1)$ continuum particles are unaffected by it. The above form of the interaction mirrors itself in the final form of the many-particle cross section; it accounts for the regular MSD scatterings in between knock-out processes as well as knock-out processes.

Part of of the aesthetic appeal and elegance of the FKK theory of nuclear scattering lies in the fact that the expression for the cross section can always be understood intuitively, since it involves convolution of single-step processes making up a multi-step process. At the end one simply adds up cross sections for all relevant multi-step processes. However, the number of of all possible multi-step processes increases rapidly as the number of continuum particles increases. While both two- and three-continuum-particle final state cross sections are computationally tractable we have attempted only the two-particle case.

The main idea underlying the expressions for n particle final state cross section is that at some point in the multistep reaction there were only $n-1$ continuum particles, any one of which could have created the nth continuum particle via the usual single-step knock-out process. Once the nth continuum particle has been created each one of the n may undergo subsequent conventional MSD scattering which does not involve creation of additional continuum particles.

In the following equations, superscripts $(1), (2), ..., (n)$ specify the final particles' ordinal numbers. All expressions are written as convolutions of cross sections rather than scattering amplitudes since this is more convenient from computational standpoint. However, in all but the initial single-step cross section the inverse S matrix squared is implicitly included to correct for the different boundary conditions of the wave-functions in the expansion of the Green's function.

The following equations are written in a recursive fashion, i.e. the (n)- particle cross section is expressed in terms of the $(n-1)$-particle cross section. The recursiveness of our results is a direct consequence of the spectator hypothesis discussed in Section 2.2.3. This spectator hypothesis allows us to account for all different pre-final MSD scatterings by a factor equal to the number of different ways each of n particles can undergo $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)}$ single step direct scatterings respectively. This is simply a binomial factor $\frac{(\mu^{(1)}+\cdots+\mu^{(n)})!}{\mu^{(1)}!\cdots\mu^{(n)}!}$.

Recursiveness is particularly useful from the computational point of view since one can use the n-particle cross section to obtain the $(n+1)$-particle cross section. We start out with a three-particle cross section which assumes the proper boundary conditions of the intermediate cross sections:

$$\begin{aligned}
\frac{d^6\sigma^{MSD}(\mathbf{k}_f^{(1)},\mathbf{k}_f^{(2)},\mathbf{k}_f^{(3)} \leftarrow \mathbf{k}_i)}{dE_f^{(1)}d\Omega_f^{(1)}dE_f^{(2)}d\Omega_f^{(2)}dE_f^{(3)}d\Omega_f^{(3)}} &= \left(\frac{m}{4\pi\hbar^2}\right)^5 \sum_{\mu^{(1)}\mu^{(2)}\mu^{(3)}=0}^{M} \frac{(\mu^{(1)}+\mu^{(2)}+\mu^{(3)})!}{\mu^{(1)}!\ \mu^{(2)}!\ \mu^{(3)}!} \\
&\int dE_y^{(1)}E_y^{(1)}d\Omega_y^{(1)} \int dE_y^{(2)}E_y^{(2)}d\Omega_y^{(2)} \int dE_y^{(3)}E_y^{(3)}d\Omega_y^{(3)} \int dE_x^{(1)}E_x^{(1)}d\Omega_x^{(1)} \int dE_x^{(2)}E_x^{(2)}d\Omega_x^{(2)} \\
&\times \left[\frac{d^2\sigma^{(\mu^{(1)})}(\mathbf{k}_f^{(1)} \leftarrow \mathbf{k}_y^{(1)})}{dE_f^{(1)}d\Omega_f^{(1)}}\right] \left[\frac{d^2\sigma^{(\mu^{(2)})}(\mathbf{k}_f^{(2)} \leftarrow \mathbf{k}_y^{(2)})}{dE_f^{(2)}d\Omega_f^{(2)}}\right] \left[\frac{d^2\sigma^{(\mu^{(3)})}(\mathbf{k}_f^{(3)} \leftarrow \mathbf{k}_y^{(3)})}{dE_f^{(3)}d\Omega_f^{(3)}}\right] \\
&\times \left[\frac{d^4\sigma^{(1)}(\mathbf{k}_y^{(2)},\mathbf{k}_y^{(3)} \leftarrow \mathbf{k}_x^{(2)})}{dE_y^{(2)}d\Omega_y^{(2)}dE_y^{(3)}d\Omega_y^{(3)}}\delta(\mathbf{k}_y^{(1)}-\mathbf{k}_x^{(1)}) + \frac{d^4\sigma^{(1)}(\mathbf{k}_y^{(1)},\mathbf{k}_y^{(3)} \leftarrow \mathbf{k}_x^{(1)})}{dE_y^{(1)}d\Omega_y^{(1)}dE_y^{(3)}d\Omega_y^{(3)}}\delta(\mathbf{k}_y^{(2)}-\mathbf{k}_x^{(2)})\right] \\
&\times \left[\frac{d^4\sigma^{MSD}(\mathbf{k}_x^{(1)},\mathbf{k}_x^{(2)} \leftarrow \mathbf{k}_i)}{dE_x^{(1)}d\Omega_x^{(1)}dE_x^{(2)}d\Omega_x^{(2)}}\right]. \qquad (2.72)
\end{aligned}$$

The above cross section can be generalized to an arbitrary number of particles in the final state,

$$
\begin{aligned}
\frac{d^{(2n)}\sigma^{MSD}(\mathbf{k}_f^{(1)},\cdots,\mathbf{k}_f^{(n)} \leftarrow \mathbf{k}_i)}{dE_f^{(1)}d\Omega_f^{(1)}\cdots dE_f^{(n)}d\Omega_f^{(n)}} &= \left(\frac{m}{4\pi\hbar^2}\right)^{(2n-1)} \sum_{\mu^{(1)}\cdots\mu^{(n)}=0}^{M} \frac{(\mu^{(1)}+\cdots+\mu^{(n)})!}{\mu^{(1)}!\cdots\mu^{(n)}!} \\
&\int dE_y^{(1)}E_y^{(1)}d\Omega_y^{(1)}\cdots\int dE_y^{(n)}E_y^{(n)}d\Omega_y^{(n)}\int dE_x^{(1)}E_x^{(1)}d\Omega_x^{(1)}\cdots\int dE_x^{(n-1)}E_x^{(n-1)}d\Omega_x^{(n-1)} \\
&\times\left[\frac{d^2\sigma^{(\mu^{(1)})}(\mathbf{k}_f^{(1)}\leftarrow\mathbf{k}_y^{(1)})}{dE_f^{(1)}d\Omega_f^{(1)}}\right]\cdots\left[\frac{d^2\sigma^{(\mu^{(n)})}(\mathbf{k}_f^{(n)}\leftarrow\mathbf{k}_y^{(n)})}{dE_f^{(n)}d\Omega_f^{(n)}}\right] \\
&\times\left[\frac{d^4\sigma^{(1)}(\mathbf{k}_y^{(1)},\mathbf{k}_y^{(n)}\leftarrow\mathbf{k}_x^{(1)})}{dE_y^{(1)}d\Omega_y^{(1)}dE_y^{(n)}d\Omega_y^{(n)}}\delta(\mathbf{k}_y^{(2)}-\mathbf{k}_x^{(2)})\cdots\delta(\mathbf{k}_y^{(n-1)}-\mathbf{k}_x^{(n-1)})+\cdots\right. \\
&\qquad\left.+\frac{d^4\sigma^{(1)}(\mathbf{k}_y^{(n-1)},\mathbf{k}_y^{(n)}\leftarrow\mathbf{k}_x^{(n-1)})}{dE_y^{(1)}d\Omega_y^{(1)}dE_y^{(3)}d\Omega_y^{(3)}}\delta(\mathbf{k}_y^{(1)}-\mathbf{k}_x^{(1)})\cdots\delta(\mathbf{k}_y^{(n-2)}-\mathbf{k}_x^{(n-2)})\right] \\
&\times\left[\frac{d^{(2n-2)}\sigma^{MSD}(\mathbf{k}_x^{(1)},\cdots,\mathbf{k}_x^{(n-1)}\leftarrow\mathbf{k}_i)}{dE_x^{(1)}d\Omega_x^{(1)}\cdots dE_x^{(n-1)}d\Omega_x^{(n-1)}}\right]
\end{aligned}
\tag{2.73}
$$

In order to simplify the above expression it is convenient to define the following quantity equal to the particle creation part of the above equation:

$$
\begin{aligned}
\frac{d^{(2n)}\sigma^{(1)}(\mathbf{k}_y^{(1)},\ldots,\mathbf{k}_y^{(n)}\leftarrow\mathbf{k}_x^{(1)},\ldots,\mathbf{k}_x^{(n-1)})}{dE_y^{(1)}d\Omega_y^{(1)}\cdots dE_y^{(n)}d\Omega_y^{(n)}} &= \\
\left[\frac{d^4\sigma^{(1)}(\mathbf{k}_y^{(1)},\mathbf{k}_y^{(n)}\leftarrow\mathbf{k}_x^{(1)})}{dE_y^{(1)}d\Omega_y^{(1)}dE_y^{(n)}d\Omega_y^{(n)}}\delta(\mathbf{k}_y^{(2)}-\mathbf{k}_x^{(2)})\cdots\delta(\mathbf{k}_y^{(n-1)}-\mathbf{k}_x^{(n-1)})+\cdots\right. \\
\left.+\frac{d^4\sigma^{(1)}(\mathbf{k}_y^{(n-1)},\mathbf{k}_y^{(n)}\leftarrow\mathbf{k}_x^{(n-1)})}{dE_x^{(1)}d\Omega_x^{(1)}dE_y^{(3)}d\Omega_y^{(3)}}\delta(\mathbf{k}_y^{(1)}-\mathbf{k}_x^{(1)})\cdots\delta(\mathbf{k}_y^{(n-2)}-\mathbf{k}_x^{(n-2)})\right]
\end{aligned}
\tag{2.74}
$$

This quantity describes a single step process taking the scattering system from $(n-1)$ continuum particles to n continuum particles via a single knock-out process. Upon utilizing the above definition, the recursion relation becomes

$$\frac{d^{(2n)}\sigma^{MSD}(\mathbf{k}_f^{(1)},\cdots,\mathbf{k}_f^{(n)}\leftarrow\mathbf{k}_i)}{dE_f^{(1)}d\Omega_f^{(1)}\cdots dE_f^{(n)}d\Omega_f^{(n)}} = \left(\frac{m}{4\pi\hbar^2}\right)^{2n-1}\sum_{\mu^{(1)}\cdots\mu^{(n)}=0}^{M}\frac{(\mu^{(1)}+\cdots+\mu^{(n)})!}{\mu^{(1)}!\cdots\mu^{(n)}!}$$
$$\int dE_y^{(1)}E_y^{(1)}d\Omega_y^{(1)}\cdots\int dE_y^{(n)}E_y^{(n)}d\Omega_y^{(n)}\int dE_x^{(1)}E_x^{(1)}d\Omega_x^{(1)}\cdots\int dE_x^{(n-1)}E_x^{(n-1)}d\Omega_x^{(n-1)}$$
$$\times\left[\frac{d^2\sigma^{(\mu^{(1)})}(\mathbf{k}_f^{(1)}\leftarrow\mathbf{k}_y^{(1)})}{dE_f^{(1)}d\Omega_f^{(1)}}\right]\cdots\left[\frac{d^2\sigma^{(\mu^{(n)})}(\mathbf{k}_f^{(n)}\leftarrow\mathbf{k}_y^{(n)})}{dE_f^{(n)}d\Omega_f^{(n)}}\right] \tag{2.75}$$
$$\times\left[\frac{d^{(2n)}\sigma^{(1)}(\mathbf{k}_y^{(1)},\ldots,\mathbf{k}_y^{(n)}\leftarrow\mathbf{k}_x^{(1)},\ldots,\mathbf{k}_x^{(n-1)})}{dE_y^{(1)}d\Omega_y^{(1)}\cdots dE_y^{(n)}d\Omega_y^{(n)}}\right]\left[\frac{d^{(2n-2)}\sigma^{MSD}(\mathbf{k}_x^{(1)},\cdots,\mathbf{k}_x^{(n-1)}\leftarrow\mathbf{k}_i)}{dE_x^{(1)}d\Omega_x^{(1)}\cdots dE_x^{(n-1)}d\Omega_x^{(n-1)}}\right]$$

Finally, we note that the bottom element in the recursion series, $\frac{d^4\sigma^{MSD}(\mathbf{k}_f^{(1)},\mathbf{k}_f^{(2)}\leftarrow\mathbf{k}_i)}{dE_f^{(1)}d\Omega_f^{(1)}dE_f^{(2)}d\Omega_f^{(2)}}$ is given by Equation (2.70).

Chapter 3

General Multistep Compound (MSC) Processes

3.1 Introduction

The multistep theory of nuclear reactions presented in Ref. [1] (FKK) has been applied to a wide variety of neutron- and proton-induced reactions for incident energies up to 250 MeV. For reviews see Refs. [2, 3]. FKK showed that the quantum statistics of multistep processes involving quasi-bound and unbound particle-hole states differ, and separated pre-equilibrium processes into two categories: multistep compound (MSC), which occurs when a chain of quasi-bound (Q) states is excited; and multistep direct MSD, which occurs when a chain of states always involving at least one unbound particle (P) is excited. This sharp separation into bound and unbound chains seemed somewhat artificial, and resulted in some criticisms of the theory, even though other quantum theories of multistep processes published after FKK also relied on this distinction. However, Nishioka, Weidenmuller, and Yoshida [4] considered the influence of MSD on transmission coefficients into the Q-chain in their multistep theory. Formal expressions were given, but no calculations were performed because of numerical difficulties in computing multistep processes in their theory.

The importance of transitions from the P to Q chains was first noted in FKK analyses of the emission spectra in 14 MeV neutron-induced reactions [5]. These processes were found to be necessary in order to obtain the correct magnitude of MSC emission. Subsequently, Ref. [6] analyzed neutron reactions on niobium between 14 and 26 MeV and showed that an accurate description of experimental emission spectra and angular distributions required the inclusion of $P \rightarrow Q$ transitions. Marcinkowski *et al.* [7, 8] also analyzed experimental emission spectra for a range of target nuclei in 20 MeV neutron-induced reactions, and emphasized the need for a gradual absorption into the compound chain. A number of other recent analyses of data have included these mechanisms [9, 10]. All these approaches estimated the strength of crossover transitions from a phenomenological phase-space model, considering the densities of P and Q states [6, 7, 8]. While such estimates were checked against the experimentally observed partitioning of MSC and MSD and found to be rather reliable [6], there is a need to justify $P \rightarrow Q$ transitions theoretically. Following the work of Nishioka *et al.*, Sato and Yoshida [11] recently investigated the influence of the imaginary part of the optical potential on transitions into the compound chain. Their results supported the predictions based upon phase-space linking, though there were indications that it might overestimate the effect. In this chapter we remove the approximation in the FKK derivation which led to the sharp separation of MSC and MSD, and we show how our new derivation naturally incorporates processes linking these two preequilibrium chains.

3.2 Feshbach-Kerman-Koonin Theory

Depending on the amount of the energy transferred to the target by the projectile, the projectile either remains an unbound continuum particle or becomes a bound particle inside the target. In the former case, the nuclear configuration after the first interaction is one particle-hole pair or (1p1h), and in the latter it is 2p1h.

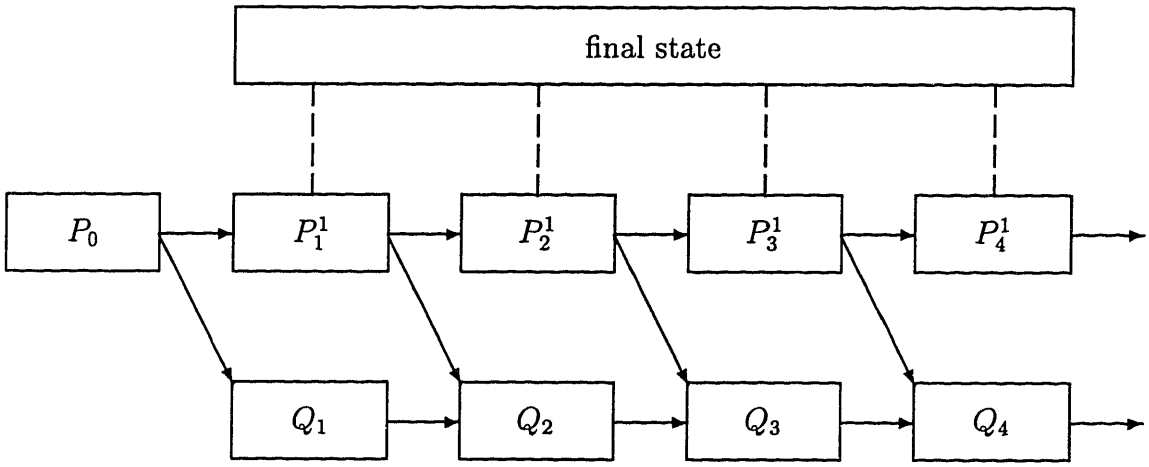

Figure 3-1: Novel MSC processes considered in this chapter are indicated by arrows from P to Q subspaces. For clarity we do not show emission from Q subspaces; the emission from Q subspaces is depicted in Fig. 2-1.

According to the original FKK, the first interaction is the only time the projectile can become bound, and if it does so it will remain bound until it is ejected to the final, outgoing state. Likewise, if it is still unbound after the first interaction it will remain unbound for the rest of the process.

By restricting itself to only this set of processes, the FKK theory omits processes in which the projectile becomes bound after undergoing one or more P chain interactions. These processes and others will be discussed in great detail later on.

3.3 MSC processes

Just like in the original FKK formulation, we begin our exposition with an expression for the MSC process $\mathcal{T}$ matrix, originally derived by Kawai, Kerman and McVoy [12]. (An insightfull and systematic derivation of this equation was done by Feshbach [23].)

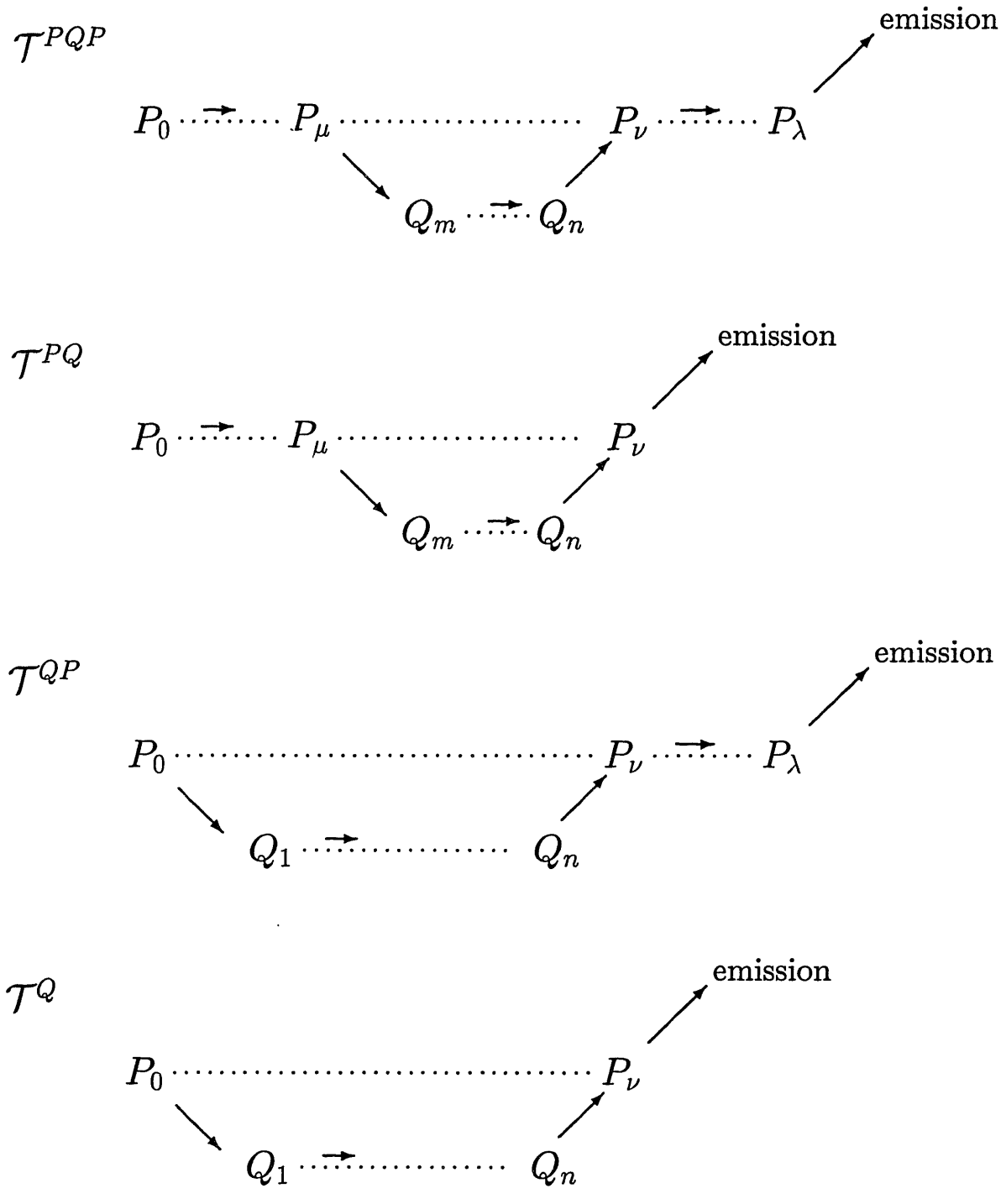

Figure 3-2: Diagrammatic representation of the four MSC processes. FKK originally considered only $\mathcal{T}^{Q}_{fi}$ processes.

$$\mathcal{T}_{fi}^{MSC} = \langle \Psi_f^{(-)} | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \Psi_i^{(+)} \rangle \tag{3.1}$$

where

$$h_{QQ} = H_{QQ} + V_{QP} \frac{1}{E^{(+)} - H_{\rm opt}} V_{PQ} \tag{3.2}$$

$$V_{PQ} = H_{PQ} \sqrt{iI/(E - H_{QQ} + iI)} \tag{3.3}$$

and I is the energy averaging interval. $\Psi_i^{(+)}$ is the distorted wave solutions of $(E - H_{\rm opt})|\Psi_i^{(+)}\rangle = 0$ with the outgoing boundary condition. As was done in [1], $H_{\rm opt}$ will be separated into its diagonal and off-diagonal components $H_{\rm opt} = H_{\rm opt}^{(D)} + v$ so that

$$|\Psi_i^{(+)}\rangle = |\phi_i^{(+)}\rangle + \frac{1}{E^{(+)} - H_{\rm opt}} v |\phi_i^{(+)}\rangle \tag{3.4}$$

In the FKK paper the above wave function $|\Psi_i^{(+)}\rangle$ was approximated by the first term i.e. $|\Psi_i^{(+)}\rangle \approx |\phi_i^{(+)}\rangle$ and likewise for $\langle \Psi_i^{(-)}|$.

We remove this approximation by inserting the complete expressions for the wave functions $|\Psi_i^{(+)}\rangle$ and $\langle \Psi_i^{(-)}|$is into (3.1). By doing so we obtain three novel terms in addition to the term that was presented in the original FKK.

$$\mathcal{T}_{fi}^{\rm MSC} = \mathcal{T}_{fi}^{Q} + \mathcal{T}_{fi}^{PQ} + \mathcal{T}_{fi}^{QP} + \mathcal{T}_{fi}^{PQP} \tag{3.5}$$

where

$$\mathcal{T}^{Q} \equiv \langle \phi_f^{(-)} | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_i^{(+)} \rangle \tag{3.6}$$

$$\mathcal{T}^{PQ} \equiv \langle \phi_f^{(-)} | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} \frac{1}{E - H_{\rm opt}} v | \phi_i^{(+)} \rangle \tag{3.7}$$

$$\mathcal{T}^{QP} \equiv \langle \phi_f^{(-)} | v \frac{1}{E - H_{\rm opt}} V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_i^{(+)} \rangle \tag{3.8}$$

$$\mathcal{T}^{PQP} \equiv \langle \phi_f^{(-)} | v \frac{1}{E - H_{\rm opt}} V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} \frac{1}{E - H_{\rm opt}} v | \phi_i^{(+)} \rangle \tag{3.9}$$

Only the first term in the expression (3.5), $\mathcal{T}^{Q}$, was analyzed in the original FKK theory (where it was denoted by $\mathcal{T}_{fi}^{MSC}$, and the other terms were omitted. $\mathcal{T}^{Q}$ describes a class of processes in which the projectile enters the Q chain via $P_0 \rightarrow Q_1$ transition, undergoes an arbitrary number of steps in the Q chain before being emitted. The second term, $\mathcal{T}^{PQ}$, describes a class of processes in which the projectile undergoes a number of P chain scatterings before entering the Q chain. The third term describes processes in which a projectile undergoes P chain steps *after* exiting the Q chain. Finally, the last term describes a process in which the projectile undergoes a number of P chain reactions before entering and after exiting the Q chain. Obviously, the common thread to processes described by all four terms is that the projectile enters the Q-chain. These processes are classed within MSC since the random phase approximation ensures that they yield angular distributions symmetric about 90^0 (once excitations take place in the Q-chain the "memory" of the initial projectile direction is always lost). Figure 3-2 shows a schematic representation of the four MSC types, with definitions of our labelings of P and Q stages.

To obtain the average of $\mathcal{T}_{fi}^{MSC}\mathcal{T}_{fi}^{MSC*}$, which is needed to obtain the cross section, we expand each of the four terms as a summation over the number of preequilibrium steps the excited nucleus undergoes before emission. For the processes originally considered by FKK, this entails summing over n, the number of steps in the Q chain, but for the new processes it brings in additional summations over the P-stage from which the Q-space is entered (μ) and the P-stage from which emission finally occurs (λ), see Fig. 3-2. Cross terms (e.g. $\langle \mathcal{T}^{PQ}\mathcal{T}^{Q*}\rangle$) vanish due to the random phase hypothesis, yielding

$$\langle|\mathcal{T}_{fi}^{MSC}|^2\rangle = \sum_{n=1}^{r}\langle|\mathcal{T}_{n}^{Q}|^2\rangle + \sum_{\mu n}^{r}\langle|\mathcal{T}_{\mu n}^{PQ}|^2\rangle + \sum_{n\lambda}^{r}\langle|\mathcal{T}_{n\lambda}^{QP}|^2\rangle + \sum_{\mu n\lambda}^{r}\langle|\mathcal{T}_{\mu n\lambda}^{PQP}|^2\rangle, \quad (3.10)$$

where

$$\mathcal{T}_{n}^{Q} = \langle\phi_f^{(-)}|V_{P_fQ_n}\frac{1}{E-h_{QQ}}V_{Q_1P_0}|\phi_i^{(+)}\rangle \quad (3.11)$$

$$\mathcal{T}_{\mu n}^{PQ} = \langle\phi_f^{(-)}|V_{P_fQ_n}\frac{1}{E-h_{QQ}}V_{Q_mP_\mu}\frac{1}{E-H_{\rm opt}}v_{P_1P_0}|\phi_i^{(+)}\rangle \quad (3.12)$$

$$\mathcal{T}_{n\lambda}^{QP} = \langle\phi_f^{(-)}|v_{P_fP_\lambda}\frac{1}{E-H_{\rm opt}}V_{P_\nu Q_n}\frac{1}{E-h_{QQ}}V_{Q_1P_0}|\phi_i^{(+)}\rangle \quad (3.13)$$

$$\mathcal{T}_{\mu n\lambda}^{PQP} = \langle\phi_f^{(-)}|v_{P_fP_\lambda}\frac{1}{E-H_{\rm opt}}V_{P_\nu Q_n}\frac{1}{E-h_{QQ}}V_{Q_mP_\mu}\frac{1}{E-H_{\rm opt}}v_{P_1P_0}|\phi_i^{(+)}\rangle \quad (3.14)$$

The meaning of the indices in the above equation is shown in Fig. 3-2. Letter r labels the equilibrium stage and for all practical purposes can be considered to be infinite.

To illustrate, $\mathcal{T}_{\mu n\lambda}^{PQP}$ describes a process in which the projectile undergoes μ MSD-type scatterings before entering the Q_m subspace (where $m = \mu, \mu \pm 1$ by chaining), follows the MSC Q-chain until the Q_n subspace, returns to the P_ν subspace (where $\nu = n, n \pm 1$ by chaining), undergoes more MSD-type rescatterings and is finally emitted from the P_λ subspace.

Obviously, $\mathcal{T}_{\nu\lambda}^{QP}$ may be viewed as a $\mu = 1$ case of $\mathcal{T}_{\mu\nu\lambda}^{PQP}$, $\mathcal{T}_{\mu\nu}^{PQ}$ may be viewed as a $\lambda = 0$ case of $\mathcal{T}_{\mu\nu\lambda}^{PQP}$, and finally $\mathcal{T}_{\nu}^{Q}$ may be viewed as a $\mu = \lambda = 0$ case of $\mathcal{T}_{\mu\nu\lambda}^{PQP}$. Propagators in Equation (3.14) will all be factorized as shown in Appendix A. To demonstrate, the $\mathcal{T}_{\mu\nu\lambda}^{PQP}$ is factorized as follows

$$\begin{aligned}\mathcal{T}_{\mu n\lambda}^{PQP} &= \langle \phi_f^{(-)} | v_{P_f P_\lambda} \; G_{P_\lambda} v_{P_\lambda P_{\lambda-1}} \cdots V_{P_\nu Q_n} G_{Q_n} V_{Q_n Q_{n-1}} \\ &\qquad \cdots G_{Q_m} V_{Q_m P_\mu} G_{P_\mu} \cdots G_{P_1} v_{P_1 P_0} | \phi_i^{(+)} \rangle. \end{aligned} \tag{3.15}$$

In deriving expressions for MSC emission, for simplicity we follow the original FKK paper in assuming spin zero projectile, ejectile and target (Ref. [2] shows how non-zero spins can be included). Also, a variety of numerical MSC calculations have shown that the MSC spectrum is very nearly isotropic for nucleon-induced reactions [6] at the energies we consider here, and therefore we shall present results only for the angle-integrated spectrum. From the above expressions for the fluctuating transition matrix we obtain the cross section for MSC emission as

3.4 $\mathcal{T}^{Q}$ processes

In this section we will review the derivation of the cross section for the MSC processes that were considered in the original FKK, i.e. processes included in the $\mathcal{T}_n^Q$ part of the $\mathcal{T}_{fi}^{MSC}$. In these processes, the incoming particle immediately enters Q-chain via a $P_i \rightarrow Q_1$ transition, undergoes damping along the Q chain and is emitted from the Q-chain without further rescattering in the P-chain.

By reviewing the derivation of these processes, we will introduce the formalism and methodology necessary to derive cross sections of more complex processes described by $\mathcal{T}^{PQ}$, $\mathcal{T}^{QP}$, and $\mathcal{T}^{PQP}$ in later sections.

As was done in deriving the cross section for the MSD processes, $\mathcal{T}^Q$ is expressed as a sum of contributions coming from different Q sub-spaces. In the following equation Q_n is a subspace from which the final particle is emitted to the final state as indicated in Fig. 3-2.

$$\mathcal{T}^Q = \sum_{n=1}^{r} \mathcal{T}_n^Q \tag{3.16}$$

where

$$\begin{aligned} \mathcal{T}_n^Q &= \langle \phi_f^{(-)} | V_{P_f Q_n} Q_n | \Psi_i^{(+)} \rangle & (3.17) \\ &= \langle \phi_f^{(-)} | V_{P_f Q_n} Q_n \frac{1}{E - h_{QQ}} V_{Q_1 P_0} | \phi_i^{(+)} \rangle & (3.18) \end{aligned}$$

We factorize the propagator as follows:

$$Q_n \frac{1}{E - h_{QQ}} Q_1 = G_{Q_n} V_{Q_n Q_{n-1}} Q_{n-1} \frac{1}{E - h_{QQ}} Q_1 \tag{3.19}$$

Next we employ a well known expression for $Q|\Psi_i^{(+)}\rangle$, a simple derivation of which can be found in [23], Chapter III Equation (2.30)[1]:

$$Q|\Psi_i^{(+)}\rangle = \frac{1}{E - h_{QQ}} V_{Q_1 P_0} |\phi_i^{(+)}\rangle \tag{3.20}$$

Multiplying the above equation by Q_n from the left we get:

$$Q_n|\Psi_i^{(+)}\rangle = Q_n \frac{1}{E - h_{QQ}} V_{Q_1 P_0} |\phi_i^{(+)}\rangle \tag{3.21}$$

We employ the chaining expression (3.19) together with 3.20 to rewrite the above equation as:

[1]The only reason the following expression is employed is to compactify the following derivation. The standard factorization of propagators that was used in the MSD derivation and presented in Appendix A would also work.

$$Q_n|\Psi_i^{(+)}\rangle = G_{Q_n} V_{Q_n,Q_{n-1}} Q_{n-1}|\Psi_i^{(+)}\rangle \tag{3.22}$$

We see that the continuation of the Q-chain is contained in $Q_{n-1}|\Psi_i^{(+)}\rangle$ of the above expression.

As shown in Appendix A, the intermediate Green's function G_{Q_n} has the following form:

$$G_{Q_n} = \frac{1}{E - h_{Q_n} - V_{Q_nQ_{n+1}} G_{Q_{n+1}} V_{Q_{n+1}Q_n}} \tag{3.23}$$

Consequently, the above Green's function is expanded in terms of a complete set of eigen-functions:

$$(h_{Q_n} + V_{Q_nQ_{n+1}} G_{Q_{n+1}} V_{Q_{n+1}Q_n})|\Psi_{Q_n\alpha}\rangle = \mathcal{E}_{Q_n\alpha}|\Psi_{Q_n\alpha}\rangle \tag{3.24}$$

where $\mathcal{E}_{Q_n\alpha}$ is a complex number if G_{n+1} is not Hermitian:

$$\mathcal{E}_{Q_n\alpha} = E_{Q_n\alpha} - i\frac{\Gamma_{Q_n\alpha}}{2} \tag{3.25}$$

In terms of the above eigenfunctions G_{Q_n} is expanded as follows:

$$G_{Q_n} = \sum_\alpha \frac{|\Psi_{Q_n\alpha}\rangle\langle\tilde{\Psi}_{Q_n\alpha}|}{E - \mathcal{E}_{Q_n\alpha}} \tag{3.26}$$

The energy dependence of the propagator G_{Q_n}, given by the term $V_{Q_nQ_{n+1}} G_{Q_{n+1}} V_{Q_{n+1}Q_n}$, has to be smooth enough for averaging to be possible. To investigate the energy dependence of this term we expand the $G_{Q_{n+1}}$ in a complete set of Q_{n+1} eigenfunctions:

$$V_{Q_nQ_{n+1}}GQ_{n+1}V_{Q_{n+1}Q_n} = \sum_\alpha V_{Q_nQ_{n+1}}|\Psi_{Q_{n+1}\alpha}\rangle \frac{1}{E-\mathcal{E}_{Q_{n+1}\alpha}}\langle\tilde{\Psi}_{Q_{n+1}\alpha}|V_{Q_{n+1}Q_n} \quad (3.27)$$

Since $\mathcal{E}_{Q_n\alpha} = E_{Q_n\alpha} - i\frac{\Gamma_{Q_n\alpha}}{2}$, the energy dependance of $V_{Q_nQ_{n+1}}G_{Q_{n+1}}V_{Q_{n+1}Q_n}$ is smooth over energy variations of the size of $\Gamma_{Q_{n+1}\alpha}$. Therefore if there are many Q_n states inside the energy interval $\Gamma_{Q_{n+1}\alpha}$ then we can energy average G_{Q_n}, knowing that $V_{Q_nQ_{n+1}}G_{Q_{n+1}}V_{Q_{n+1}Q_n}$ does not have a rapid energy dependence in the averaging region. The requirement that there be many Q_n states in the energy region $\Gamma_{n+1\alpha}$ is equivalent to the statement:

$$\Gamma_{Q_{n+1}} \gg D_{Q_n} \quad (3.28)$$

where $D_{Q_n} = 1/\rho_{Q_n}$ is the spacing of energy levels in Q_n. We proceed by inserting the Green's function expansion (3.26) in the expression for $\mathcal{T}_n^Q$ in Equation (3.18).

$$\mathcal{T}_n^Q = \sum_\alpha \langle\phi_f^{(-)}V_{PQ_n}\Psi_{Q_n\alpha}\rangle \frac{1}{E-\mathcal{E}_{Q_n\alpha}}\langle\tilde{\Psi}_{Q_n\alpha}V_{Q_n,Q_{n-1}}\Psi_i^{(+)}\rangle \quad (3.29)$$

and proceed to evaluate the average of $|\mathcal{T}_{fi}^Q|^2$:

$$\langle|T_{fi}^{Q}|^2\rangle = \langle\sum_{n,n'} T_n^Q T_{n'}^{Q*}\rangle \tag{3.30}$$

$$= \sum_n \langle T_n^Q T_n^{Q*}\rangle \tag{3.31}$$

$$= \sum_{\alpha\alpha'} \langle\langle\phi_f^{(-)} V_{PQ_n}\Psi_{Q_n\alpha}\rangle \frac{1}{E-\mathcal{E}_{Q_n\alpha}} \langle\tilde{\Psi}_{Q_n\alpha} V_{Q_n,Q_{n-1}}\Psi_i^{(+)}\rangle$$
$$\times \langle\phi_f^{(-)} V_{PQ_n}\Psi_{Q_n\alpha'}\rangle^* \frac{1}{E-\mathcal{E}^*_{Q_n\alpha'}} \langle\tilde{\Psi}_{Q_n\alpha'} V_{Q_n,Q_{n-1}}\Psi_i^{(+)}\rangle^*\rangle \tag{3.32}$$

$$\langle|T_n^Q|^2\rangle = \sum_\alpha \langle|\langle\phi_f^{(-)} V_{PQ_n}\Psi_{Q_n\alpha}\rangle|^2 \frac{1}{|E-\mathcal{E}_{Q_n\alpha}|^2} |\langle\tilde{\Psi}_{Q_n\alpha} V_{Q_n,Q_{n-1}}\Psi_i^{(+)}\rangle|^2\rangle \tag{3.33}$$

The summation over α is then transformed into an energy integral over $(E - W/2, E + W/2)$ where $W \gg (\Gamma_{Q_n} \gg D_{Q_n})$ to assure that many (overlapping) resonances are included in the summation. Conversion from summation to integration introduces an additional factor of energy density of states which is simply the inverse of level spacing in Q_n, $\rho_{Q_n} = 1/D_{Q_n}$.

$$\langle|T_n^Q|^2\rangle = \langle\int_{E-W/2}^{E+W/2} dE |\langle\phi_f^{(-)} V_{PQ_n}\Psi_{Q_n\alpha}\rangle|^2 \frac{\rho_{Q_n}}{|E-\mathcal{E}_{Q_n\alpha}|^2} |\langle\tilde{\Psi}_{Q_n\alpha} V_{Q_n,Q_{n-1}}\Psi_i^{(+)}\rangle|^2\rangle \tag{3.34}$$

As was the case in the MSD derivation, the integration over energy in the above expression is done assuming that the energy variation of $|\langle\phi_f^{(-)} V_{PQ_n}\Psi_{Q_n\alpha}\rangle|^2$ is sufficiently smooth that it may be replaced by its average value in the integration region. The average is evaluated with respect to Q_n states $\Psi_{Q_n\alpha}$ in the energy region $(E - W/2, E + W/2)$. Thus, the integration is performed over the denominator only and the numerator is replaced by its average over the integration region:

$$\sum_{\alpha} \frac{1}{|E-\mathcal{E}_{Q_n\alpha}|^2} = \int_{E_n-W/2}^{E_n+W/2} \frac{\rho_{Q_n}}{(E-E_{Q_n})^2+(\Gamma_{Q_n}/2)^2} dE$$
$$= \frac{2}{\Gamma_{Q_n} D_{Q_n}} \arctan(x)\Big|_{x=-W/\Gamma_{Q_n}}^{x=W/\Gamma_{Q_n}}$$
$$\approx \frac{2\pi}{\Gamma_{Q_n} D_{Q_n}} \tag{3.35}$$

The above integral was evaluated in the limit $W \gg \Gamma_{Q_n}$ which is necessary for averaging to take place. In the above integral Γ_{Q_n} is, rigorously speaking, a function of energy $\Gamma_{Q_n}(E)$ but over the region of integration it may be replaced by its average value which we simply denote as Γ_n. Thus, Equation (3.34) can be recast as:

$$\langle|\mathcal{T}_n^Q|^2\rangle = \frac{\Gamma_n^{(f)}}{\Gamma_n} \frac{\langle|\langle\tilde{\Psi}_{Q_n\alpha} V_{Q_n,Q_{n-1}} \Psi_i^{(+)}\rangle|^2\rangle}{D_n} \tag{3.36}$$

where

$$\Gamma_n^{(f)} \equiv 2\pi\langle|\langle\phi_f^{(-)} V_{PQ_n} \Psi_{Q_n}\rangle|^2\rangle \tag{3.37}$$

is the escape width averaged over Q_n states included in summation over α. One then uses chaining hypothesis (3.22) to express $Q_{n-1}|\Psi_i^{(+)}\rangle$ in terms of $Q_{n-2}|\Psi_i^{(+)}\rangle$ and the process is repeated:

$$\langle|\mathcal{T}_n^Q|^2\rangle = \frac{\Gamma_n^{(f)}}{\Gamma_n} \langle\sum_{\beta} |\langle\Psi_{Q_n} V_{Q_nQ_{n-1}} \Psi_{Q_{n-1}\beta}\rangle|^2 \frac{1}{|E-\mathcal{E}_{Q_{n-1}\beta}|^2} |\langle\tilde{\Psi}_{Q_{n-1}\beta} V_{Q_{n-1},Q_{n-2}} \Psi_i^{(+)}\rangle|^2\rangle \tag{3.38}$$

By performing energy integration and employing all assumptions as in the previous step we obtain

$$\langle|\mathcal{T}_n^Q|^2\rangle = \frac{\Gamma_n^{(f)}}{\Gamma_n}\frac{\Gamma_{n-1}^{\downarrow}}{\Gamma_{n-1}}\frac{\langle|\langle\tilde{\Psi}_{Q_{n-1}\beta}V_{Q_{n-1},Q_{n-2}}\Psi_i^{(+)}\rangle|^2\rangle}{D_{n-1}} \tag{3.39}$$

where

$$\Gamma_{n-1}^{\downarrow} \equiv \frac{2\pi}{D_n}\langle|\langle\tilde{\Psi}_{Q_n\alpha}V_{Q_n,Q_{n-1}}\Psi_{Q_{n-1}\beta}\rangle|^2\rangle \tag{3.40}$$

is referred to as the spreading width and is averaged over labels α and β. The above procedure is iterated until reaching the incoming wave-function. The final expression can be written as:

$$\begin{aligned}\langle|\mathcal{T}_{fi}^Q|^2\rangle &= \sum_{n=1}^{r}\langle|\mathcal{T}_n^Q|^2\rangle \\ &= \frac{1}{2\pi^2}\sum_{n=1}\frac{\Gamma_n^{\uparrow(f)}}{\Gamma_n}\left[\prod_{k=1}^{n-1}\frac{\Gamma_k^{\downarrow}}{\Gamma_k}\right]\frac{2\pi\Gamma_1{}^{(i)}}{D_1},\end{aligned} \tag{3.41}$$

where

$$\Gamma_1^{(i)} \equiv 2\pi\langle|\langle\tilde{\Psi}_{Q_1\alpha}V_{Q_1,P_0}\phi_i^{(+)}\rangle|^2\rangle \tag{3.42}$$

The final expression for the cross section is obtained by using the following form of the cross section:

$$\frac{d\langle\sigma^Q\rangle}{dU} = \sum_n\frac{d\langle\sigma_n^Q\rangle}{dU} = \frac{4\pi^3}{k_i^2}\sum_n\langle|\mathcal{T}_n^Q|^2\rangle \tag{3.43}$$

The above expression for the cross section in terms of $|\mathcal{T}_{fi}|^2$, unlike the equivalent expression that was used in the MSD formalism Equation (2.19), assumes that the wave-functions are normalized to the square-root of state density, i.e. $\Psi\Psi^* = \rho$. Upon summing over n we get:

$$\frac{d\langle\sigma^Q\rangle}{dU} = \frac{\pi}{k_i^2}\sum_{n=1}^{r}\sum_{\nu=n-1}^{n+1}\frac{\langle\Gamma^{\uparrow}_{n\nu}(U)\rangle}{\Gamma_n}\left[\prod_{k=1}^{n-1}\frac{\Gamma^{\downarrow}_k}{\Gamma_k}\right]\frac{2\pi\Gamma_1{}^{(i)}}{D_1}, \tag{3.44}$$

where the escape width describing emission via the P_ν channel in the above equation is given by

$$\Gamma^{\uparrow}_{n\nu}(U) = 2\pi\,\langle|\langle\chi^{(-)}_{\nu f}V_{P_\nu Q_n}\Psi_{Q_n}\rangle|^2\rangle \tag{3.45}$$

The average in the above equation is to be taken over wave-functions in Q_n and P_ν.

3.5 $\mathcal{T}^{PQ}$ processes

To evaluate the cross section for processes whose $\mathcal{T}_{fi}$-matrix elements are given by (3.12) we factorize the P-space propagator in (3.12) as follows:

$$\begin{aligned} P_\mu\frac{1}{E^{(+)}-H_{\rm opt}}P_1 &= G_{P_\mu}v_{P_\mu P_{\mu-1}}P_{\mu-1}\frac{1}{E^{(+)}-H_{\rm opt}}P_1 \\ &= G_{P_\mu}v_{P_\mu P_{\mu-1}}G_{P_{\mu-1}}v_{P_{\mu-1}P_{\mu-2}}\cdots G_{P_{\nu+2}}v_{P_{\nu+2}P_{\nu+1}}G_{P_1} \end{aligned} \tag{3.46}$$

where

$$G_{P_\mu} = \frac{1}{E^{(+)}-H^{(D)}_{P_\mu}-v_{P_\mu P_{\mu+1}}G_{P_{\mu+1}}v_{P_{\mu+1}P_\mu}} \tag{3.47}$$

The wavefunctions in which the above propagator will be expanded are given by:

$$(H^{(D)}_{P_\mu}+v_{P_\mu P_{\mu+1}}G_{P_{\mu+1}}v_{P_{\mu+1}P_\mu})|\Psi_{P_\mu\alpha}\rangle = (\frac{\hbar}{2m}k^2_\mu+\mathcal{E}_{P_\mu\alpha})|\Psi_{P_\mu\alpha}\rangle \tag{3.48}$$

Similarly, we factorize the Q chain propagator as follows:

$$Q_n \frac{1}{E - h_{QQ}} Q_m = G_{Q_n} V_{Q_n,Q_{n-1}} G_{Q_{n-1}} V_{Q_{n-1},Q_{n-2}} \cdots G_{Q_{m+2}} V_{Q_{m+2},Q_{m+1}} G_{Q_m} \quad (3.49)$$

Instead of considering the most general formula, we shall evaluate the simplest $P \rightarrow Q$ transition process, $P_0 \rightarrow P_1 \rightarrow Q_2 \rightarrow P_3 \rightarrow$ *emission*, from which the general case can be induced. The $\mathcal{T}_{fi}^{PQ}$ matrix element for this process is

$$\mathcal{T}_{fi}^{PQ} = \langle \phi_f^{(-)} | V_{P_3 Q_2} G_{Q_2} V_{Q_2 P_1} G_{P_1} v_{P_1 i} | \phi_i^{(+)} \rangle \quad (3.50)$$

where the Green's functions are then decomposed in terms of the corresponding eigenfunctions as follows:

$$\begin{aligned} G_{P_1} &= \sum_\alpha \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{|\Psi_{P_1\alpha}^{(+)}\rangle\langle\tilde{\Psi}_{P_1\alpha}^{(+)}|}{E^{(+)} - \frac{\hbar^2}{2m}k_1^2 - \mathcal{E}_{P_1\alpha}} \\ G_{Q_2} &= \sum_\beta \frac{|\Psi_{Q_2\beta}^{(+)}\rangle\langle\tilde{\Psi}_{Q_2\beta}^{(+)}|}{E - \mathcal{E}_{Q_2\beta}} \end{aligned} \quad (3.51)$$

The process described by (3.50) turns out to be a very important one because it provides a mechanism by which Q_1 is bypassed. As is known from extensive numerical applications of FKK, most of the MSC preequilibrium emission comes from Q_1 and consequently if Q_1 is bypassed the MSC emission will be diminished. This is a welcome development because the theory overpredicts the MSC emission if only $\mathcal{T}_{fi}^{Q}$ are taken into account, as can be seen in Figures (3-4) and (3-5).

We proceed by taking the square of (3.50) and expanding the Green's functions of the resulting expression as in (3.51):

$$G_{Q_2}V_{Q_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v^*_{iP_1}G^*_{P_1}V^*_{P_1Q_2}G^*_{Q_2} = \sum_{\beta\beta'\alpha\alpha'}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}'_1}{(2\pi)^3}$$
$$\times\frac{1}{E-\mathcal{E}_{Q_2\beta}}\bar{v}_{\beta\alpha}(Q_2,\mathbf{k}_1)\frac{1}{E^{(+)}-(\hbar/2m)k_1^2-\mathcal{E}_{P_1\alpha}}\bar{v}_{\alpha i}(\mathbf{k}_1,\mathbf{k}_i)$$
$$\times\bar{v}_{i\alpha'}(\mathbf{k}_i,\mathbf{k}'_1)\frac{1}{E^{(-)}-(\hbar/2m)k_1'^2-\mathcal{E}_{P_1\alpha'}}\bar{v}_{\alpha'\beta'}(\mathbf{k}'_1,Q_2)\frac{1}{E-\mathcal{E}_{Q_2\beta'}} \quad (3.52)$$

where

$$\bar{v}_{\alpha i}(\mathbf{k}_1,\mathbf{k}_i) = \langle\tilde{\Psi}^{(+)}_\alpha(\mathbf{k}_1)|v|\phi^{(+)}(\mathbf{k}_i)\rangle \quad (3.53)$$

$$\bar{v}_{\beta\alpha}(Q_2,\mathbf{k}_1) = \langle\Psi_{Q_2\beta}|v|\Psi^{(+)}_\alpha(\mathbf{k}_1)\rangle \quad (3.54)$$

α and β in the above equation label nuclear states in P_1 and Q_2, respectively.

Integrations over intermediate momenta $\mathbf{k}_1$, $\mathbf{k}'_1$, and summations over α, α' in (3.52) are dealt with identically as in derivation of the MSD cross section (Equations (2.23)- (2.30)). First, the RPA hypothesis is used to reduce the double summation over α and α' to a single summation over α. Then, the summation over α is translated into an energy integral and all other steps used in derivation of MSD are repeated to obtain:

$$G_{Q_2}V_{Q_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v^*_{iP_1}G^*_{P_1}V^*_{P_1Q_2}G^*_{Q_2} =$$
$$\sum_{\beta\beta'}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{1}{E-\mathcal{E}_{Q_2\beta}}\bar{v}_{\beta 1}(Q_2,\mathbf{k}_1)\bar{v}^*_{1\beta'}(\mathbf{k}_1,Q_2)\frac{1}{E-\mathcal{E}^*_{Q_2\beta'}}$$
$$\times 2\pi^2\rho_{1p1h}(U_1)\rho(k_1)|\bar{v}_{1i}(\mathbf{k}_1,\mathbf{k}_i)|^2 \quad (3.55)$$

By comparing the above equation with (2.33) we can see that the bottom line of the equation above is equivalent to the single step direct amplitude. Employing this observation together with elimination of summation over β' via RPA, we obtain:

$$G_{Q_2}V_{Q_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v_{iP_1}^*G_{P_1}^*V_{P_1Q_2}^*G_{Q_2}^* = \sum_\beta \int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{|\bar{v}_{\beta 1}(Q_2,\mathbf{k}_1)|^2}{|E-\mathcal{E}_{Q_2\beta}|^2}\frac{d^2w^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dE_1 d\Omega_1} \tag{3.56}$$

Then we perform the summation over β. This summation can be performed in the same way as in the case of $\mathcal{T}_{fi}^{Q}$ processes (see equations (3.33)-(3.35) and the surrounding discussion) to obtain:

$$G_{Q_2}V_{Q_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v_{iP_1}^*G_{P_1}^*V_{P_1Q_2}^*G_{Q_2}^* = \int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{2\pi\Gamma_1^{(1)}}{D_1}\left[\frac{d^2w^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dE_1 d\Omega_1}\right] \tag{3.57}$$

We can see that the above expression is simply a convolution of a single step direct amplitude and a strength function for entering Q_2 subspace from P_1. The difference between the above strength function and the one in (3.42) is that the incoming momentum here is $\mathbf{k}_1$, not $\mathbf{k}_i$, as indicated by its superscript " $^{(1)}$ ". Furthermore $\mathbf{k}_1$ is an integration variable and a reflection of the fact that the projectile may lose an arbitrary amount of energy in the direct step prior to entering Q chain. Thus the following definition of the strength function for entering Q_2 with momentum $\mathbf{k}_i$ seems natural:

$$\frac{2\pi\Gamma_2^{(i)}}{D_2} \equiv G_{Q_2}V_{Q_2P_1}G_{P_1}v_{P_1i}|\phi_i^{(+)}\rangle\langle\tilde{\phi}_i^{(+)}|v^*_{iP_1}G^*_{P_1}V^*_{P_1Q_2}G^*_{Q_2} \tag{3.58}$$

$$= \int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{2\pi\Gamma_1^{(1)}}{D_1}\left[\frac{d^2w^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dE_1 d\Omega_1}\right] \tag{3.59}$$

We generalize the above expression to an arbitrary number of P chain scatterings before entering the Q chain. In what follows, P_μ is the P subspace from which $Q_m = Q_{\mu+1}$ is entered. The ensuing expression for the corresponding strength function is then

$$\frac{2\pi\Gamma_\mu{}^{(i)}}{D_\mu} = \int \frac{d\mathbf{k}_1}{(2\pi)^3}\cdots\frac{d\mathbf{k}_\mu}{(2\pi)^3}\frac{2\pi\Gamma_1{}^{(\mu)}}{D_1}\left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_\mu \leftarrow \mathbf{k}_{\mu-1})}{dU_\mu d\Omega_\mu}\right]$$
$$\times\left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_{\mu-1} \leftarrow \mathbf{k}_{\mu-2})}{dU_{\mu-1}d\Omega_{\mu-1}}\right]\cdots\left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right] \tag{3.60}$$

The multistep compound scattering which occurs after the Q_m stage is identical to that already described in derivation of pure MSC processes, described in Section 3.4. Consequently we can employ the cross section for $\mathcal{T}_{fi}^{Q}$ processes, Equation (3.44), to describe the Q-chain damping and the emission part of a $\mathcal{T}_{fi}^{PQ}$ process.

In order to obtain the cross section for $\mathcal{T}_{fi}^{PQ}$ processes, we add contributions coming from P to Q transitions at all [2] possible stages (i.e. contributions from $P_1 \rightarrow Q_2, P_2 \rightarrow Q_3$, etc. transitions). This, however, can be accomplished by a simple modification of the expression for pure MSC processes (3.44). Subscript 1 of the initial strength function in (3.44) is replaced by μ so that the new strength function is now given by (3.60):

$$\frac{2\pi\Gamma_1{}^{(i)}}{D_1} \rightarrow \frac{2\pi\Gamma_\mu{}^{(i)}}{D_\mu} \tag{3.61}$$

Furthermore, the product of depletion factors has to be modified so that the first subspace in the Q-chain cascade is Q_μ:

[2] P_0 to Q_1 transition process is not included in this summation; it is accounted for in $\mathcal{T}_{fi}^{Q}$

$$\left[\prod_{k=1}^{n-1}\frac{\Gamma_{kJ}^{\downarrow}}{\Gamma_{kJ}}\right]\rightarrow\left[\prod_{k=\mu}^{n-1}\frac{\Gamma_{kJ}^{\downarrow}}{\Gamma_{kJ}}\right] \tag{3.62}$$

so that the cross section for processes described by $\mathcal{T}_{fi}^{PQ}$ is obtained by adding contributions from all μ's:

$$\frac{d\langle\sigma^{PQ}\rangle}{dU}=\frac{\pi}{k_i^2}\sum_{\mu=2}^{r}\sum_{n=\mu}^{r}\sum_{\nu=n-1}^{n+1}\frac{\langle\Gamma_{n\nu}^{\uparrow}(U)\rangle}{\Gamma_n}\left[\prod_{k=\mu}^{n-1}\frac{\Gamma_k^{\downarrow}}{\Gamma_k}\right]\frac{2\pi\Gamma_\mu{}^{(i)}}{D_\mu}, \tag{3.63}$$

3.6 $\mathcal{T}^{QP}$ processes

As was done in derivation of the cross section for $\mathcal{T}^{PQ}$ processes, we will utilize the cross section for $\mathcal{T}^{Q}$ processes (3.44) to describe the $P_0\rightarrow Q_1$ entry into the Q chain and the subsequent Q chain cascade.

From the last subspace in the Q chain cascade, Q_n, the particle returns to the P chain[3] at P_ν and undergoes additional MSD scattering along the P-chain until the P_λ subspace. The final state is to be projected out of the three P subspaces connected to P_λ via the chaining hypothesis. The projection of the final state, as well as all other aspects of treating the P-chain cascade are the same as that for the usual MSD processes.

Now we have all the necessary ingredients to write the cross section for processes described by $\mathcal{T}_{fi}^{QP}$:

$$\frac{d\langle\sigma^{QP}\rangle}{dU}=\frac{\pi}{k_i^2}\sum_{n=1}^{r}\sum_{\nu=n-1}^{n+1}\sum_{\lambda=\nu+1}^{r}\frac{\langle\Gamma_{n\nu}^{\uparrow\lambda}(U)\rangle}{\Gamma_n}\left[\prod_{k=1}^{n-1}\frac{\Gamma_k^{\downarrow}}{\Gamma_k}\right]\frac{2\pi\Gamma_1{}^{(i)}}{D_1}, \tag{3.64}$$

where

[3]The transition from Q_n to $P_\nu, \nu = n-1, n, n+1$ is simply an escape width (3.45).

$$\Gamma^{\uparrow\lambda}_{n\nu}(U_f) = \int \frac{d\mathbf{k}_\lambda}{(2\pi)^3} \cdots \frac{d\mathbf{k}_\nu}{(2\pi)^3} \left[\frac{d^2w^{(1)}(\mathbf{k}_f \leftarrow \mathbf{k}_\lambda)}{dU_f d\Omega_f}\right] \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_\lambda \leftarrow \mathbf{k}_{\lambda-1})}{dU_\lambda d\Omega_\lambda}\right] \cdots$$
$$\times \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_{\nu+2} \leftarrow \mathbf{k}_{\nu+1})}{dU_{\nu+2} d\Omega_{\nu+2}}\right] \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_{\nu+1} \leftarrow \mathbf{k}_\nu)}{dU_{\nu+1} d\Omega_{\nu+1}}\right] \tilde{\Gamma}^{\uparrow}_{n\nu}(E_\nu) \qquad (3.65)$$

is the modified escape width.

To make the final expression (3.64) resemble the original FKK expression even more we define a new $\Gamma^{\uparrow}_{n\nu}$ which accounts for all possible rescattering down the P chain,

$$\Gamma'^{\uparrow}_{n\nu} = \sum_{\lambda=\nu}^{r} \Gamma^{\lambda\uparrow}_{n\nu} \qquad (3.66)$$

In effect this simply hides the summation over λ in (3.69).

Summation over λ can be caried out as follows:

$$\sum_{\lambda=\nu} \Gamma^{\lambda\uparrow}_{n'} = \frac{m}{4\pi^2\hbar^2} \int E_{\nu'} dE_{\nu'} \left[\frac{dw^{MSD}(E_f \leftarrow E_{\nu'})}{dE_f}\right] \Gamma^{\uparrow}_{n\nu}(E_{\nu'}) \qquad (3.67)$$

In the above equation we have made explicit use of the fact that the emission width is spherically symmetric for all practical purposes.

3.7 $\mathcal{T}^{PQP}$ processes

Having found cross sections for the two simpler classes of processes $\mathcal{T}^{PQ}$ and $\mathcal{T}^{QP}$, given by (3.63) and (3.64) respectively, we can induce the cross section for $\mathcal{T}^{PQP}$ processes.

For the $\mathcal{T}^{PQP}$ processes both the entrance strength function and the escape width have to be modified to account for P chain scattering before entering and after exiting the Q-chain:

$$\frac{d\langle\sigma^{PQP}\rangle}{dU} = \frac{\pi}{k_i^2}\sum_{\mu=2}^{n}\sum_{n=\mu}^{r}\sum_{\nu=n-1}^{n+1}\sum_{\lambda=\nu+1}^{r}\frac{\langle\Gamma_{n\nu}^{\uparrow\lambda}(U)\rangle}{\Gamma_n}\left[\prod_{k=\mu}^{n-1}\frac{\Gamma_k^{\downarrow}}{\Gamma_k}\right]\frac{2\pi\Gamma_\mu{}^{(i)}}{D_\mu}, \tag{3.68}$$

3.8 Summary of the MSC Theoretical Results

To obtain the total MSC cross section we add the cross section for $\mathcal{T}_{fi}^{Q}$, $\mathcal{T}_{fi}^{PQ}$, $\mathcal{T}_{fi}^{QP}$ and $\mathcal{T}_{fi}^{PQP}$ processes given by (3.44), (3.63), (3.64) and (3.68):

$$\frac{d\langle\sigma^{MSC}\rangle}{dU} = \pi\lambda^2\sum_J(2J+1)\sum_{\mu=1}^{r}\sum_{n=\mu}^{r}\sum_{\nu=n-1}^{n+1}\sum_{\lambda=\nu}^{r}\sum_l\frac{\langle\Gamma_{nJ}^{\uparrow\nu\lambda l}(U)\rangle}{\Gamma_{nJ}}\left[\prod_{k=\mu}^{n-1}\frac{\Gamma_{kJ}^{\downarrow}}{\Gamma_{kJ}}\right]\frac{2\pi\Gamma_\mu{}^{(i)}}{D_\mu}, \tag{3.69}$$

where U is the residual nucleus energy, J is the composite nucleus angular momentum, l is the emitted particle angular momentum, $\langle\Gamma_{nJ}^{\uparrow\nu\lambda l}(U)\rangle$ is the emission width of stage n, and Γ_{nJ} is the total width of stage n. The quantity in the square brackets is the depletion factor, which accounts for flux lost to emission from previous stages, and $2\pi\Gamma_\mu/D_\mu{}^{(i)}$ is the strength function for entrance into Q-space from stage μ.

This equation looks very similar to that of FKK (see, for instance, Eq. (3.41) in Ref. [1]). But in addition to the sum over preequilibrium stages n, ν already present in the FKK equation, there are summations over: (1) μ, the initial Q-stage entered; (2) λ, the final P-stage from which emission occurs. Physically, initial MSD scatterings in P-space are included via the strength function, which is defined for each stage of entrance into Q-space (see below). Likewise, subsequent MSD scatterings are included within the definitions of the emission rates, which include the label λ (see below). In the special case of μ=0 and λ=ν, Eq. (3.69) reduces to the original FKK Eq. (3.41) [1], while the other terms in the summations account for linking between MSC and MSD. The above equation is a generalization of the expression given in the appendix of Marcinkowski *et al.*'s paper [8].

As in the original FKK paper, the random phase approximation results in a coherence only between continuum particles with the same direction, resulting in the well-known convolution structure of multistep transitions in the P-chain. This enables the strength function for a given stage μ to be expressed as a convolution of 1-step MSD probabilities, finally folded into the entrance strength function. Thus, the μth entrance strength function, which describes $P \rightarrow Q$ processes, is given by

$$\frac{2\pi\Gamma_\mu{}^{(i)}}{D_\mu} = \int \frac{d\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d\mathbf{k}_\mu}{(2\pi)^3} \frac{2\pi\Gamma_1{}^{(\mu)}}{D_1} \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_\mu \leftarrow \mathbf{k}_{\mu-1})}{dU_\mu d\Omega_\mu}\right] \times \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_{\mu-1} \leftarrow \mathbf{k}_{\mu-2})}{dU_{\mu-1} d\Omega_{\mu-1}}\right] \cdots \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right], \tag{3.70}$$

where the terms in the square brackets are exactly the same 1-step probabilities that are used in the MSD theory. Likewise, we describe transitions in P-space that can follow MSC emission as

$$\Gamma_{nJ}^{\uparrow\nu\lambda l} = \int \frac{d\mathbf{k}_\lambda}{(2\pi)^3} \cdots \frac{d\mathbf{k}_\nu}{(2\pi)^3} \left[\frac{d^2w^{(1)}(\mathbf{k}_f \leftarrow \mathbf{k}_\lambda)}{dU_f d\Omega_f}\right] \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_\lambda \leftarrow \mathbf{k}_{\lambda-1})}{dU_\lambda d\Omega_\lambda}\right] \cdots \times \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_{\nu+2} \leftarrow \mathbf{k}_{\nu+1})}{dU_{\nu+2} d\Omega_{\nu+2}}\right] \left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_{\nu+1} \leftarrow \mathbf{k}_\nu)}{dU_{\nu+1} d\Omega_{\nu+1}}\right] \Gamma_{nJ}^{\uparrow\nu l}(E_\nu) \tag{3.71}$$

where $\Gamma_{nJ}^{\uparrow\nu l}(E_\nu)$ is the width for immediate emission. Our numerical calculations of additional rescatterings in P-space after emission using Eq. (3.71) indicate that they are of minor importance. But as we shall show below, $P \rightarrow Q$ transitions [from Eq. (3.60)] can be very significant. The entrance function $2\pi\Gamma_1/D_1$ was calculated microscopically using the approach of Bonetti *et al.* [2], using constant wavefunctions [1].

In summary, we have generalized the FKK theory to include a linking of the P and Q chains which results in three new types of MSC emission. Our work provides a theoretical basis for the presence of processes usually described in the literature by the phase-space model [6]. Without this linking, the FKK theory overestimates preequilibrium emission. Our theory gives an improved description of measurements, but there is still some excess compared to experimental data. Further improvement may be obtained with more sophisticated calculations.

3.9 Applications and Numerical Results

3.9.1 Converting Amplitudes to Cross Sections

It is computationally advantageous to express a transition probability $\frac{d^2w^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dE_1 d\Omega_1}$ in terms of a corresponding single step cross section:

$$\frac{d^2\omega^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dE_1 d\Omega_1} = \frac{\pi\hbar^2 k_i}{m} \frac{d^2\sigma^{(1)}(E_1, \Omega_1 \leftarrow E_i, \Omega_0)}{dE_1 d\Omega_1}. \tag{3.72}$$

which upon substitution into (3.59) and conversion of integration variables from momentum to energy gives:

$$\frac{2\pi\Gamma_2{}^{(i)}}{D_2} = \frac{mc^2\sqrt{E_i}}{4\pi^2(\hbar c)^2} \int dE_1 \sqrt{E_1} \int d\Omega_1 \frac{d^2\sigma^{(1)}(E_1, \Omega_1 \leftarrow E_i, \Omega_0)}{d\Omega_1 dE_1} \frac{2\pi\Gamma_1{}^{(1)}}{D_1}. \tag{3.73}$$

The strength function for a transition at a later stage is given by:

$$\frac{2\pi\Gamma_n{}^{(i)}}{D_n} = \frac{mc^2}{4\pi^2(\hbar c)^2} \int dE_1 E_1 \int d\Omega_1 \frac{d^2\sigma^{(1)}(E_1, \Omega_1 \leftarrow E_i, \Omega_0)}{d\Omega_1 dE_1} \frac{2\pi\Gamma_{n-1}{}^{(1)}}{D_{n-1}}. \tag{3.74}$$

The angular integral can be done to reduce the above convolution expression to:

$$\frac{2\pi\Gamma_n{}^{(i)}}{D_n} = \frac{mc^2}{4\pi^2(\hbar c)^2}\int dE_1 E_1 \left[\frac{d\sigma^{(1)}(E_1 \leftarrow E_i)}{dE_1}\right] \frac{2\pi\Gamma_{n-1}{}^{(1)}}{D_{n-1}}. \tag{3.75}$$

3.9.2 The Initial Strength Function

Finally we define the initial strength function

$$\begin{aligned}\frac{2\pi\Gamma_1{}^{(i)}}{D_1} &= (2\pi)^2\omega(2,1,E_i)\sum_{Q,j_3}\left[(2Q+1)(2j_3+1)\right. \\ &\left.\times F(Q)R_1(j_3)\begin{pmatrix} j & j_3 & Q \\ 0 & 0 & 0\end{pmatrix} I_l^2\right]\end{aligned} \tag{3.76}$$

where

$$F(Q) = \sum_{j_1,j_2}(2j_1+1)R_1(j_1)(2j_2+1)R_1(j_2)\begin{pmatrix} j_1 & j_2 & Q \\ 0 & 0 & 0\end{pmatrix}^2 \tag{3.77}$$

$R_1(j_1)$ is the Gaussian angular momentum distribution (2.38) and $\omega(p,h,E)$ is the particle-hole density of states given by Williams formula (2.37).

For the overlap integral I_l^2, we used the FKK prescription [1]:

$$I_l^2 = \frac{1}{2\pi}\frac{4r_0^3V_0^2kmT_l}{\hbar^2A} \tag{3.78}$$

where T_l is a transition coefficient at a particular energy.

We adjusted V_0 so that the sum of cross sections for all possible $P\rightarrow Q$ transitions adds up to the total multistep compound cross section $\sigma^{MSC} = \sigma_R - \sigma^{MSD}$

$$\frac{\pi}{k_i^2}\sum_l (2l+1)\sum_{n=1}\frac{2\pi\Gamma_{nl}{}^{(i)}}{D_n} = \sigma_R(E_i) - \sigma^{MSD} \tag{3.79}$$

3.9.3 Numerical Results

We apply our linked MSD-MSC formalism to describe 14 MeV neutron inelastic scattering on niobium, which has become a test-case in the literature for theoretical analyses. The computational approach and input parameters described in Ref. [6] were used in the FKK-GNASH code system [6, 13] to obtain the cross sections. Unlike Ref. [6] which estimated the crossover transitions using phase space arguments, we explicitly calculate them with Eqs. (3.60,3.71). Also, when calculating the multistep processes we use modified DWBA matrix elements, which we denote as "MDW" (sometimes called "non-normal" DWBA in the literature). The MDW matrix elements differ from their corresponding normal DWBA matrix elements by an inverse S-matrix factor. They use the boundary conditions that naturally appear in the complete set of states inserted in the evaluation of the intermediate state optical Green's function, and have been advocated by Kawai and others, and noted recently in Koning and Akkerman's derivation [14].

The residual nucleon-nucleon interaction strength $V_0^{\rm MSD}$ (which affects the magnitude of MSD emission as well as the multistep P-space transitions involved in $P\rightarrow Q$ processes) is treated as a parameter in the theory. We obtained $V_0^{\rm MSD}$=36 MeV by fitting the MSD emission cross section extracted from experimental data [6]. The interaction strength for MSC processes, which enters the overlap integral in the entrance strength function, was obtained from unitarity since the total $P\rightarrow Q$ flux equals the reaction minus the MSD cross section.

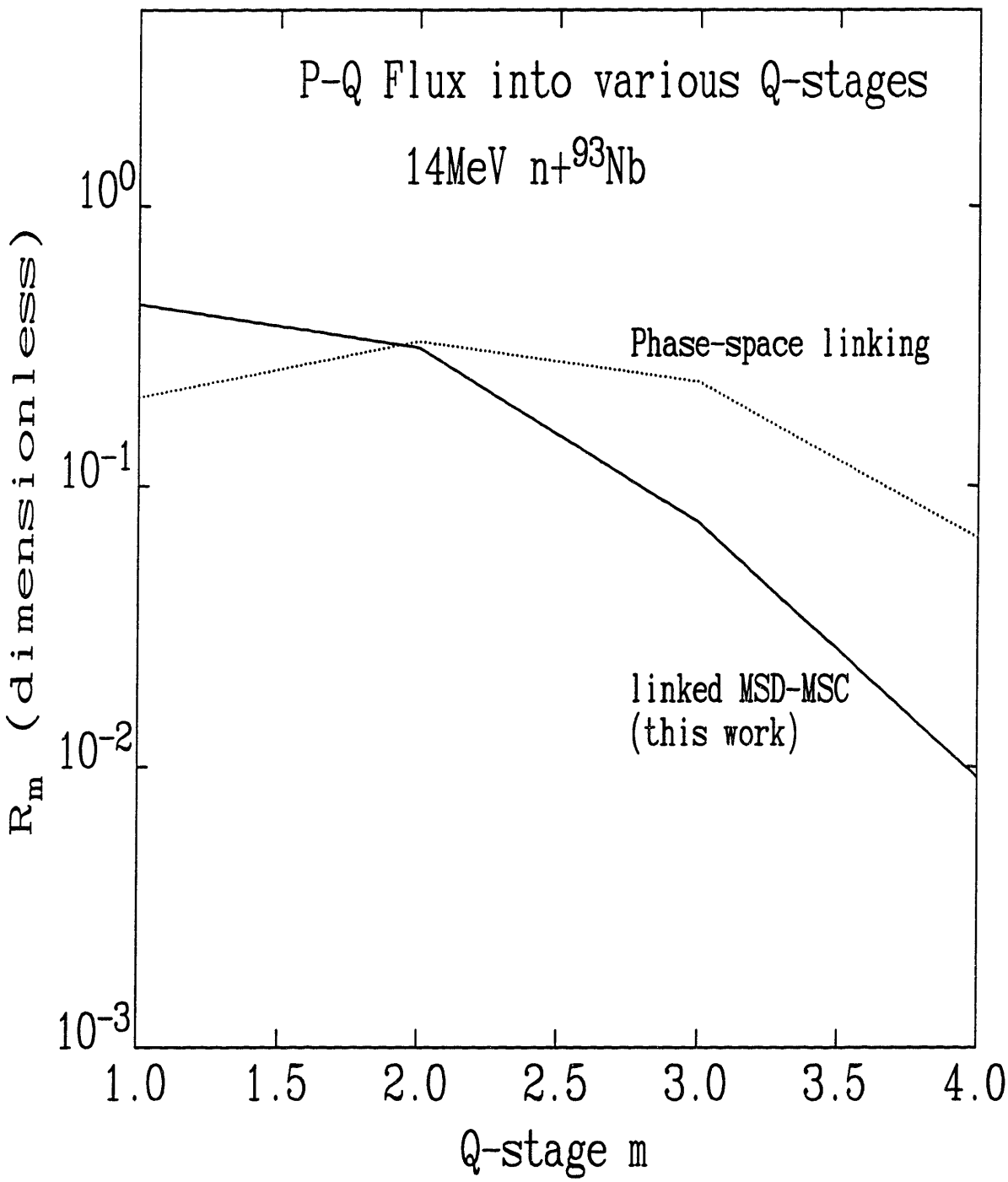

Figure 3-3: Fraction of reaction flux entering the Q-chain at stage m.

In Fig. 3-3 we show our results for the $P \rightarrow Q$ flux into the compound chain at the various stages m. For comparison we also show the phenomenological phase-space model [6, 8] results. R_m denotes the fraction of the reaction cross section that enters the compound chain at stage m.

It is evident that $P \rightarrow Q$ transitions beyond the initial $P_0 \rightarrow Q_1$ are significant and absorption into the compound chain is a gradual process. Even though the phase-space estimate does not agree exactly with our theoretical calculations, there is a qualitative agreement in so far as both predict a gradual absorption. Our calculation of the initial $P_0 \rightarrow Q_1$ flux is about twice that predicted by the phase-space model, and in good accord with the results of Sato and Yoshida [11]. Our theory predicts an increasing importance of $P \rightarrow Q$ crossover transitions with increasing incident energy. Indeed, when the incident energy exceeds the sum of the Fermi and binding energies, absorption into the compound chain always take place after a number of P-space transitions.

The angle-integrated emission spectrum from our linked MSD-MSC theory in the 14 MeV ^{93}Nb(n, n') reaction is shown in Fig.3-4. The contributions of MSC, MSD, and Hauser-Feshbach processes are indicated. Given that the complete spectrum, and the MSD-MSC linking, is obtained quantum mechanically without any parameter adjustment, the theory describes the measurements rather well. If the theory predicted an even slower absorption into the compound chain the high-energy MSC emission would be further reduced, improving the agreement with data in the 7-10 MeV range. This is because the dominant contribution to MSC comes from the first $2p1h$ stage (Q_1), and bypassing this stage reduces MSC emission. For comparison, we show in Fig. 3-5 the spectrum when the MSD-MSC linking is achieved using the phenomenological phase space model [8], with the MDW prescription for MSD emission. This model predicts an even smaller MSC emission (due to the large amount of $P \rightarrow Q$ transitions as shown in Fig.3-3) and describes the data well.

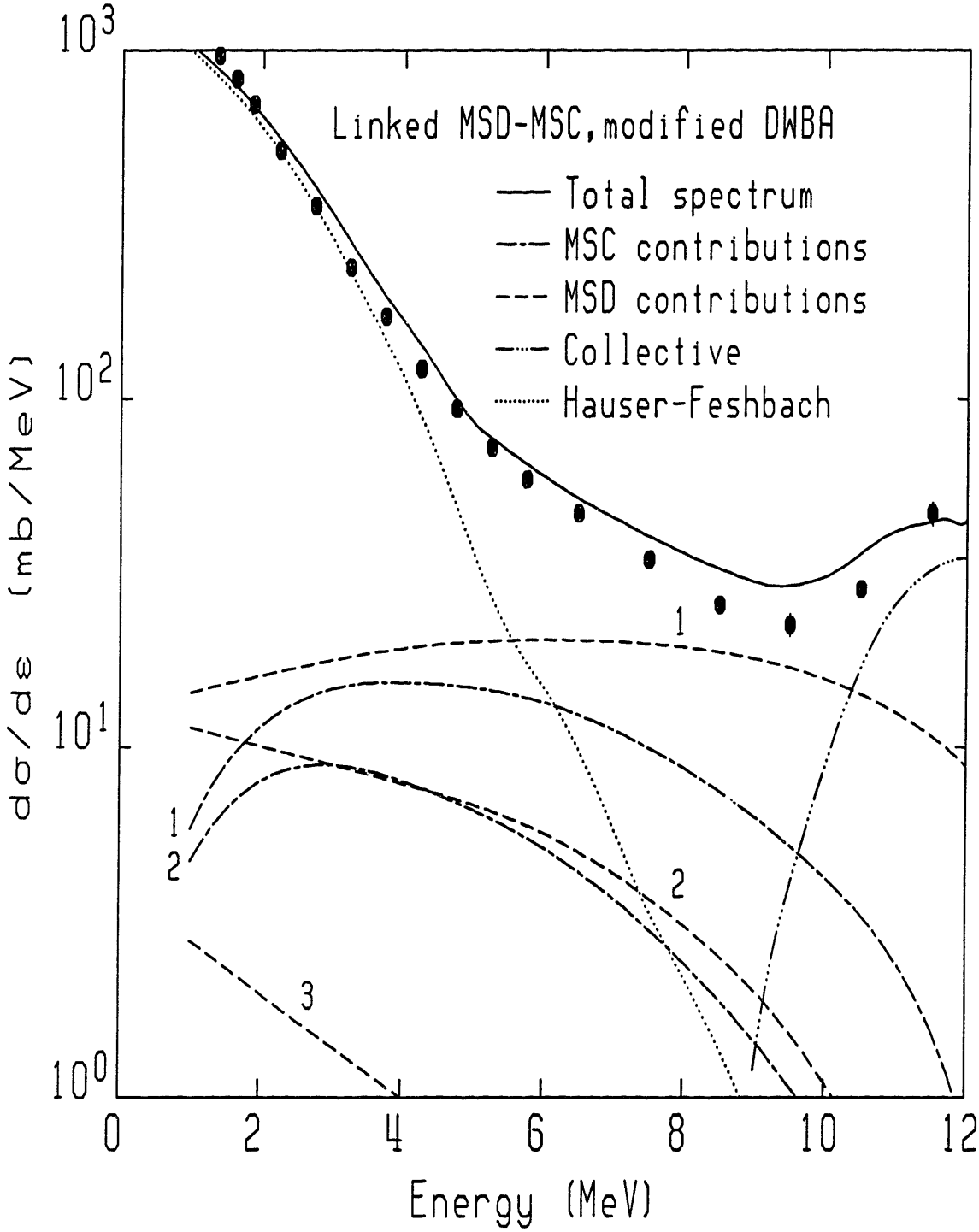

Figure 3-4: Linked MSD-MSC theory compared with angle-integrated 14 MeV ^{93}Nb(n, n') data of Pavlik *et al.* [15] The labels 1,2,3 on the MSD and MSC curves describe contributions from the different preequilibrium stages P1, P2, P3 and Q1, Q2, Q3 respectively.

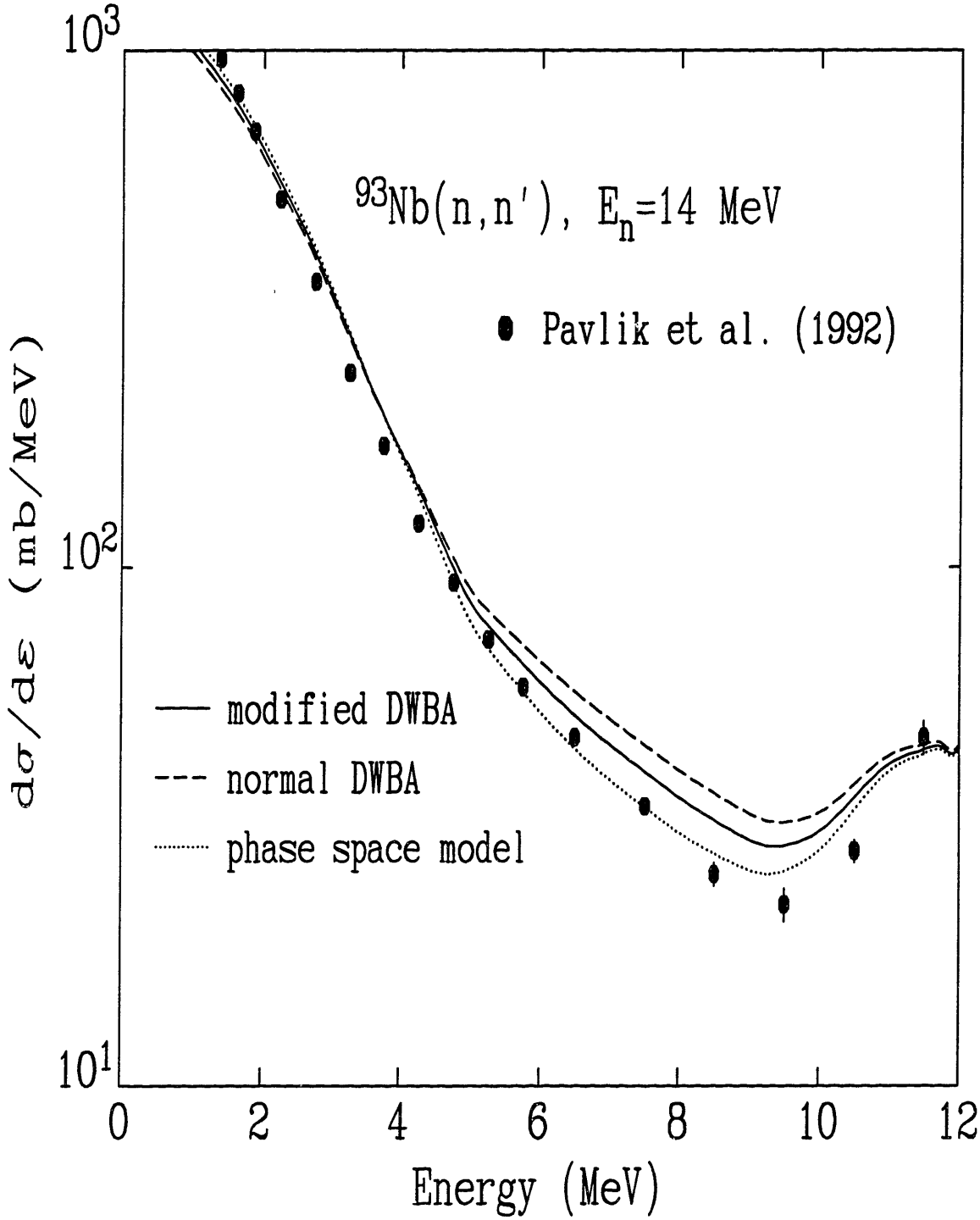

Figure 3-5: Linked MSD-MSC theory compared with angle-integrated 14 MeV ^{93} Nb (n, n') data of Pavlik *et al.* [15]. Results obtained with the present theory are shown for both the modified DWBA (MDW) and normal DWBA boundary conditions. Also shown is the result obtained with the phenomenological phase-space model. With increased linking between the P and Q chains, the initial MSC Q-stage tends to be bypassed and MSC emission decreases.

For comparison we also performed the linked MSD-MSC theory calculations using the prescription proposed by Feshbach [16] which uses normal DWBA matrix elements in the multistep calculations. But we found that this approach provides a negligibly small amount of $P \rightarrow Q$ transitions (since multistep processes in the P-chain are significantly smaller here) and results in a preequilibrium emission spectrum that largely overpredicts experiment (see Fig. 3-5). Further details of our experience in using MDW and normal DWBA are given in Ref. [17]. Marcinkowski's article [8] concluded by highlighting one issue – why should the multistep processes involved in $P \rightarrow Q$ transitions be significant if the multistep contributions to MSD emission are very small? Our work addresses this question. We only find significant $P \rightarrow Q$ transitions beyond the initial $P_0 \rightarrow Q_1$ when the multistep MSD contributions are also significant, i.e. when MDW is used.

Chapter 4

Finding a Single-step cross section by Deconvolution

4.1 Introduction

Disagreement has long existed over whether the multistep direct cross section (2.34) is a convolution of DWBA matrix elements as advocated by Feshbach [16], or a convolution of modified DWBA elements (intermediate steps) and a DWBA element (the final step) as advocated by Kerman [39]. In this chapter we will describe how intermediate single step cross section can be found by fitting the total MSD cross section to experimental data . Comparison of the intermediate step amplitude and the final step amplitude obtained by this algorithm reveals whether the intermediate steps are DWBA or not.

Since the MSD cross section is a sum of convolutions of single-step direct cross sections, the problem of finding a single step cross section is highly nonlinear and non-local. It is non-local because the value of the single-step cross section for a pair of incoming and outgoing momenta affects the total MSD cross section at every other pair of incoming and outgoing momenta.

4.2 Formalism

For simplicity, we will represent cross sections simply as $\sigma(\mathbf{k}_{\text{out}}, \mathbf{k}_{\text{in}})$ and will drop numerical constants. To start out, we assume that only the last cross section in the convolution, $\sigma^{(1)}(\mathbf{k}_f, \mathbf{k}_\mu)$, has the DWBA boundary conditions. Single step cross sections with modified DWBA matrix elements have been denoted by $\tilde{\sigma}$

Thus we write the total MSD cross section as

$$\begin{aligned} \sigma^{MSD}(\mathbf{k}_f, \mathbf{k}_i) &= \sum_\mu \int d\mathbf{k}_\mu \cdots d\mathbf{k}_1 \; \sigma^{(1)}(\mathbf{k}_f, \mathbf{k}_\mu) \; \tilde{\sigma}^{(1)}(\mathbf{k}_{\mu-1}, \mathbf{k}_{\mu-2}) \cdots \; \tilde{\sigma}^{(1)}(\mathbf{k}_1, \mathbf{k}_i) \\ &= \sum_\mu \sigma^{(\mu)} \end{aligned} \tag{4.1}$$

For future purposes we also define the following quantity

$$\begin{aligned} \tilde{\sigma}^{MSD}(\mathbf{k}_f, \mathbf{k}_i) &= \sum_\mu \int d\mathbf{k}_\mu \cdots d\mathbf{k}_1 \; \tilde{\sigma}^{(1)}(\mathbf{k}_f, \mathbf{k}_\mu) \; \tilde{\sigma}^{(1)}(\mathbf{k}_{\mu-1}, \mathbf{k}_{\mu-2}) \cdots \; \tilde{\sigma}^{(1)}(\mathbf{k}_1, \mathbf{k}_i) \\ &= \sum_\mu \tilde{\sigma}^{(\mu)} \end{aligned} \tag{4.2}$$

We will show how a single-step cross section can be found by fitting theoretical σ^{MSD} to an experimentally measured σ^{exp}. The numerical method we employ is a derivative of the so called imaginary time method in which equations are solved by minimizing a certain quantity. Usually one minimizes the total energy of the system, but in this case we will be minimizing the χ^2 defined as:

$$\chi^2 = \int d\mathbf{k}_f d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_f, \mathbf{k}_i) - \sigma^{\exp}(\mathbf{k}_f, \mathbf{k}_i))^2}{\triangle(\mathbf{k}_f, \mathbf{k}_i)} \tag{4.3}$$

where $\triangle(\mathbf{k}_f, \mathbf{k}_i)$ is experimental uncertainty of $\sigma^{\exp}$for a particular initial $\mathbf{k}_i$ and final $\mathbf{k}_f$ momenta. The χ^2 is then varied with respect to $\sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})$ and $\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})$ where $\mathbf{k}_{\rm out}$ and $\mathbf{k}_{\rm in}$ represent the point in phase-space at which $\sigma^{(1)}$ or $\tilde{\sigma}^{(1)}$ is varied.

In the imaginary-time method, we start out by setting the initial single step cross sections to arbitrary functions $\sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})_{t=0}$ and $\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})_{t=0}$. The single step cross sections are iteratively updated as follows:

$$\sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})_{t+\delta t} = \sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})_t - \frac{\delta\chi^2}{\delta\sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})}\,\delta t \tag{4.4}$$

$$\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})_{t+\delta t} = \tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})_t - \frac{\delta\chi^2}{\delta\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})}\,\delta t \tag{4.5}$$

where δt is the iteration period. In practice, δt is assigned the largest value that still makes this method give a convergent solution. Choosing δt in such a manner minimizes the amount of computation required.

Next we show how to evaluate $\frac{\delta\chi^2}{\delta\sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})}$ and $\frac{\delta\chi^2}{\delta\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})}$. Functional differentiation with respect to $\sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})$ is carried out utilizing

$$\frac{\delta\sigma^{(1)}(\mathbf{k}'_{\rm out}, \mathbf{k}'_{\rm in})}{\delta\sigma^{(1)}(\mathbf{k}_{\rm out}, \mathbf{k}_{\rm in})} = \delta(\mathbf{k}'_{\rm out} - \mathbf{k}_{\rm out})\,\delta(\mathbf{k}'_{\rm in} - \mathbf{k}_{\rm in}) \tag{4.6}$$

so that

$$\frac{\delta\chi^2}{\delta\sigma^{(1)}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})} = 2\int d\mathbf{k}_f d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_i)-\sigma^{\rm exp}(\mathbf{k}_f,\mathbf{k}_i))}{\triangle(\mathbf{k}_f,\mathbf{k}_i)} \sum_{\mu=1}^{\infty} \int d\mathbf{k}_\mu \cdots d\mathbf{k}_1\, \delta(\mathbf{k}_f-\mathbf{k}_{\rm out})\, \delta(\mathbf{k}_\mu-\mathbf{k}_{\rm in})\, \tilde{\sigma}^{(1)}(\mathbf{k}_\mu,\mathbf{k}_{\mu-1})\cdots \cdots\, \tilde{\sigma}^{(1)}(\mathbf{k}_1,\mathbf{k}_i) \tag{4.7}$$

Next, we perform integration over $d\mathbf{k}_\mu$ and employ the definition of $\tilde{\sigma}^{MSD}(\mathbf{k}_{\rm in},\mathbf{k}_i)$ in (4.2) to rewrite the above equation as

$$\frac{\delta\chi^2}{\delta\sigma^{(1)}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})} = 2\int d\mathbf{k}_f d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_i)-\sigma^{\rm exp}(\mathbf{k}_f,\mathbf{k}_i))}{\triangle(\mathbf{k}_f,\mathbf{k}_i)} \times[\, \delta(\mathbf{k}_f-\mathbf{k}_{\rm out})\, \delta(\mathbf{k}_{\rm in}-\mathbf{k}_i) + \delta(\mathbf{k}_f-\mathbf{k}_{\rm out})\tilde{\sigma}^{MSD}(\mathbf{k}_{\rm in},\mathbf{k}_i)] \tag{4.8}$$

Finally we perform the integration over $d\mathbf{k}_f$ yielding

$$\frac{\delta\chi^2}{\delta\sigma^{(1)}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})} = 2\int d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_{\rm out},\mathbf{k}_i)-\sigma_{\rm exp}(\mathbf{k}_{\rm out},\mathbf{k}_i))}{\triangle(\mathbf{k}_{\rm out},\mathbf{k}_i)} \times[\, \delta(\mathbf{k}_{\rm in}-\mathbf{k}_i) + \tilde{\sigma}^{MSD}(\mathbf{k}_{\rm in},\mathbf{k}_i)] \tag{4.9}$$

Next, we perform the differentiation procedure with respect to $\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})$.

$$\frac{\delta\chi^2}{\delta\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})} = 2\int d\mathbf{k}_f d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_i)-\sigma^{\rm exp}(\mathbf{k}_f,\mathbf{k}_i))}{\triangle(\mathbf{k}_f,\mathbf{k}_i)} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\mu-1} \int d\mathbf{k}_\mu \cdots d\mathbf{k}_\nu d\mathbf{k}_{\nu-1}\cdots d\mathbf{k}_1\, \sigma^{(1)}(\mathbf{k}_f,\mathbf{k}_\mu)\, \tilde{\sigma}^{(1)}(\mathbf{k}_\mu,\mathbf{k}_{\mu-1})\cdots \cdots\tilde{\sigma}^{(1)}(\mathbf{k}_{\nu+1},\mathbf{k}_\nu)\, \delta(\mathbf{k}_\nu-\mathbf{k}_{\rm out})\, \delta(\mathbf{k}_{\nu-1}-\mathbf{k}_{\rm in})\, \tilde{\sigma}^{(1)}(\mathbf{k}_{\nu-1},\mathbf{k}_{\nu-2})\cdots \cdots\, \tilde{\sigma}^{(1)}(\mathbf{k}_1,\mathbf{k}_i) \tag{4.10}$$

Integration over $d\mathbf{k}_\nu$ and $d\mathbf{k}_{\nu-1}$ is carried out to yield:

$$\frac{\delta\chi^2}{\delta\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})} = 2\int d\mathbf{k}_f d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_i)-\sigma^{\rm exp}(\mathbf{k}_f,\mathbf{k}_i))}{\triangle(\mathbf{k}_f,\mathbf{k}_i)} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\mu} \int d\mathbf{k}_\mu \cdots d\mathbf{k}_{\nu+1} d\mathbf{k}_{\nu-2} \cdots d\mathbf{k}_1 \; \sigma^{(1)}(\mathbf{k}_f,\mathbf{k}_\mu)\; \tilde{\sigma}^{(1)}(\mathbf{k}_\mu,\mathbf{k}_{\mu-1}) \cdots \cdots \tilde{\sigma}^{(1)}(\mathbf{k}_{\nu+1},\mathbf{k}_{\rm out})\; \tilde{\sigma}^{(1)}(\mathbf{k}_{\rm in},\mathbf{k}_{\nu-2}) \cdots \; \tilde{\sigma}^{(1)}(\mathbf{k}_1,\mathbf{k}_i) \tag{4.11}$$

Finally we employ the definitions of $\tilde{\sigma}^{MSD}(\mathbf{k}_{\rm in},\mathbf{k}_i)$ and $\sigma^{MSD}(\mathbf{k}_{\rm in},\mathbf{k}_i)$ in (4.2) and (4.1), respectively so that summations in the above equation are hidden inside those definitions.

$$\begin{aligned}\frac{\delta\chi^2}{\delta\tilde{\sigma}^{(1)}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})} &= 2\int d\mathbf{k}_f d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_i)-\sigma^{\rm exp}(\mathbf{k}_f,\mathbf{k}_i))}{\triangle(\mathbf{k}_f,\mathbf{k}_i)} \\ &\quad [\,\sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_{\rm out})\,\delta(\mathbf{k}_{\rm in}-\mathbf{k}_i) + \sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_{\rm out})\,\tilde{\sigma}^{MSD}(\mathbf{k}_{\rm in},\mathbf{k}_i)] \\ &= 2\int d\mathbf{k}_f d\mathbf{k}_i \frac{(\sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_i)-\sigma^{\rm exp}(\mathbf{k}_f,\mathbf{k}_i))}{\triangle(\mathbf{k}_f,\mathbf{k}_i)} \\ &\quad \sigma^{MSD}(\mathbf{k}_f,\mathbf{k}_{\rm out})[\,\delta(\mathbf{k}_{\rm in}-\mathbf{k}_i) + \tilde{\sigma}^{MSD}(\mathbf{k}_{\rm in},\mathbf{k}_i)]\end{aligned} \tag{4.12}$$

4.3 Evaluation of Convolution Integrals

Next we show how to evaluate the $\sigma^{MSD}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})$ and $\tilde{\sigma}^{MSD}(\mathbf{k}_{\rm out},\mathbf{k}_{\rm in})$. First we rewrite $\sigma^{(\mu)}$ and $\tilde{\sigma}^{(\mu-1)}$ in (4.1) and (4.2) in a recursive fashion

$$\sigma^{(\mu)}(\mathbf{k}_\mu,\mathbf{k}_i) = \int d\mathbf{k}_{\mu-1}\; \sigma^{(1)}(\mathbf{k}_\mu,\mathbf{k}_{\mu-1})\; \tilde{\sigma}^{\mu-1}(\mathbf{k}_{\mu-1},\mathbf{k}_i) \tag{4.13}$$

$$\tilde{\sigma}^{(\mu-1)}(\mathbf{k}_{\mu-1},\mathbf{k}_i) = \int d\mathbf{k}_{\mu-2}\; \tilde{\sigma}^{(1)}(\mathbf{k}_{\mu-1},\mathbf{k}_{\mu-2})\; \tilde{\sigma}^{\mu-2}(\mathbf{k}_{\mu-2},\mathbf{k}_i) \tag{4.14}$$

The advantage of writing $\sigma^{(\mu)}$ and $\tilde{\sigma}^{(\mu-1)}$ in a recursive way is that one has to deal with only one integration over intermediate momenta at a time. In what follows we will describe how to perform the integration over intermediate momenta for $\sigma^{(\mu)}$. The integration involved in evaluating $\tilde{\sigma}^{(\mu)}$ is completely identical.

To perform the above integration we transform variables from momenta to energy and angles.

$$\begin{aligned}\sigma^{(\mu)}(E_\mu, \theta_\mu \leftarrow E_i) &= \int dE_{\mu-1} d\cos\theta_{\mu-1} d\phi_{\mu-1} \\ &\times \sigma^{(1)}[E_\mu, \theta_\mu \leftarrow E_{\mu-1}, \theta_{\mu-1}, \phi_{\mu-1}] \\ &\times \tilde{\sigma}^{(\mu-1)}[E_{\mu-1}, \theta_{\mu-1}, \phi_{\mu-1} \leftarrow E_i] \end{aligned} \tag{4.15}$$

where E_μ, θ_μ and ϕ_μ are energy and angles of a particle's direction after μ-th scattering in the coordinate system whose z-axis is fixed by the initial direction of the projectile. However, since a single step cross section is a function of the angle between the incoming and outgoing angles for each step, it is useful to rewrite the above equation in terms of relative angles denoted by primed variables:

$$\begin{aligned}\sigma^{(\mu)}(E_\mu, \theta_\mu \leftarrow E_i) &= \sum_\mu \int dE_{\mu-1} d\cos\theta_{\mu-1} d\phi_{\mu-1} \\ &\times \sigma^{(1)}[E_\mu, \theta'_\mu(\theta_\mu, \theta_{\mu-1}, \phi_{\mu-1}) \leftarrow E_{\mu-1}] \\ &\times \tilde{\sigma}^{(\mu-1)}[E_{\mu-1}, \theta_{\mu-1} \leftarrow E_i] \end{aligned} \tag{4.16}$$

Relative angles denoted by prime have been written as functions of angles in the fixed frame. In the last line of the above equation we have taken advantage of the freedom to fix the azimuthal angle as $\phi_\mu = 0$.

The relative angle can be found in terms of absolute angles by application of the cosine theorem. We are looking for the angle between two unit vectors defined by their Cartesian coordinates in the frame whose z-axis is given by the direction of the projectile:

$$\begin{aligned} \hat{\Omega}_{\mu-1} &= (\sin\theta_{\mu-1}\cos\phi_{\mu-1},\ \sin\theta_{\mu-1}\sin\phi_{\mu-1},\ \cos\theta_{\mu-1}) \\ \hat{\Omega}_{\mu} &= (\sin\theta_{\mu}\cos\phi_{\mu},\ 0,\ \cos\theta_{\mu}) \end{aligned} \tag{4.17}$$

We can find the cosine of the relative angle between any two vectors by taking a dot product between them.

$$\hat{\Omega}_{\mu}\cdot\hat{\Omega}_{\mu-1} = \cos\theta'_{\mu} \tag{4.18}$$

so that the cosine of the angle between $\hat{\Omega}_{\mu-1}$ and $\hat{\Omega}_{\mu}$, $\cos\theta'_{\mu}$ is given by

$$\cos\theta'_{\mu} = \sin\theta_{\mu-1}\sin\theta_{\mu}\cos\phi_{\mu} + \cos\theta_{\mu-1}\cos\theta_{\mu} \tag{4.19}$$

Thus we have defined all elements necessary to perform numerical computation of a single step scattering based on experimentally observed cross section. The generalization to multi particle processes is straight forward and we plan to implement it in the near future. [1]

[1] We have applied our deconvolution program on a 1-dimensional test problem and were able to satisfactorily find a single step cross section from the multi-step cross section.

Chapter 5

Multistep Distorted Wave Impulse Approximation

5.1 Introduction

In an attempt to extend the FKK theory and the work in this thesis to higher energies we show how the optical potential v-matrix elements $\bar{v}_{\beta\alpha}(\mathbf{k}', \mathbf{k}) = \langle \tilde{\Psi}_{\beta}^{(+)}(\mathbf{k}')|v|\Psi^{(+)}(\mathbf{k})\rangle$ that enter the calculation of the MSD cross sections can be expressed in terms of t-matrix elements. The structure of the MSD cross sections remains intact. We use only the first order optical potential formalism in calculation of the v-matrix elements in terms of transition matrix. Multistep t-matrix formalism is known to successfully describe nucleon-nucleus scattering at energies of several hundreds MeV [22].

5.2 Establishing a Connection

A simple way to find the optical potential for the FKK multi-step direct scattering appropriate at energies of a few hundred MeV is to exploit a well known result of the formalism developed by Kerman, McManus, and Thaler [22] which relates optical potential to the averaged nucleon-nucleon t-matrix. The averaged nucleon-nucleon t-matrix accounts for the fact that the bound nucleon is in the nuclear medium. For simplicity, in the following expression we write only the first order optical potential although the optical potential has been evaluated to the second order in t in [22]:

$$v^{(1)} = \langle \Psi_f | \sum_{i=1}^{A} t_i(E) | \Psi_0 \rangle + O(\frac{1}{A}) \tag{5.1}$$

where $t_i(E)$ is a transition matrix element for scattering of the projectile over the i-th bound nucleon and $\langle \Psi_f |$ and $|\Psi_0\rangle$ are the final and the initial target nucleus wavefunctions. For the processes we will consider, $|\Psi_0\rangle$ is always the ground nuclear state and $\langle \Psi_f |$ is either a particle-hole or a single hole nuclear state. Which of the two possibilities for $\langle \Psi_f |$ occurs depends on the amount of energy deposited by the projectile nucleon.

At sufficiently high energies $t_i(E)$ can be approximated by the free nucleon-nucleon transition matrix $t_i^{\text{free}}(E)$ which neglects the fact that the bound nucleon is in the nuclear medium. This approximation is called the Distorted Wave Impulse Approximation or DWIA. A computational advantage of using DWIA is that $t_i^{\text{free}}(E)$ does not require averaging over nuclear configurations, which was so crucial in determining the MSD single step cross section (2.33). The averaging over nuclear configurations is not necessary in DWIA because the $t_i^{\text{free}}(E)$ is, by definition, independent of the nuclear configuration.

Upon inserting nuclear wave functions and rearranging integrals we simplify the expression for $v^{(1)}$ at a more innocent looking expression:

$$
\begin{aligned}
v^{(1)} &= \sum_{i=1}^{A} \int \frac{d\mathbf{p}_1}{2\pi^3}\frac{d\mathbf{p}_2}{2\pi^3}\cdots\frac{d\mathbf{p}_i}{2\pi^3}\frac{d\mathbf{p}'_i}{2\pi^3}\cdots\frac{d\mathbf{p}_A}{2\pi^3} \\
&\quad \cdot\Phi^{\dagger}_{1p1h}(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}'_i,\ldots,\mathbf{p}_A)\langle\mathbf{p}'_i|t_i(E)|\mathbf{p}_i\rangle \\
&\quad \cdot\Phi_0(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_i,\ldots,\mathbf{p}_A) \qquad (5.2) \\
&= \int \Omega_{f0}(\mathbf{p}',\mathbf{p})\langle\mathbf{p}'|t(E)|\mathbf{p}\rangle\frac{d\mathbf{p}}{2\pi^3}\frac{d\mathbf{p}'}{2\pi^3} \qquad (5.3)
\end{aligned}
$$

where in the last equation we made the following substitution:

$$
\begin{aligned}
\Omega_{f0}(\mathbf{p}',\mathbf{p}) &= \sum_{i=1}^{A} \int \frac{d\mathbf{p}_1}{2\pi^3}\frac{d\mathbf{p}_2}{2\pi^3}\cdots\frac{d\mathbf{p}_{i-1}}{2\pi^3}\frac{d\mathbf{p}_{i+1}}{2\pi^3}\cdots\frac{d\mathbf{p}_A}{2\pi^3} \\
&\quad \cdot\Phi^{\dagger}_{1p1h}(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{i-1},\mathbf{p}',\mathbf{p}_{i+1}\cdots\mathbf{p}_A) \\
&\quad \cdot\Phi_0(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{i-1},\mathbf{p},\mathbf{p}_{i+1}\cdots\mathbf{p}_A) \qquad (5.4)
\end{aligned}
$$

If $v^{(1)}$ is identified as the off-diagonal part v of the optical Hamiltonian $H_{\text{opt}} = H^{(D)} + v$ (2.5) which was used in the FKK multi-step direct formalism, then we can employ the above formulae to rewrite the FKK MSD cross section in terms of the t-matrix. Since the potential v of the FKK formalism appears only in combinations of the form $\langle\tilde{\Psi}(\mathbf{k}')^{(+)}|v|\Psi(\mathbf{k})^{(+)}\rangle$ [1], we evaluate $v^{(1)}$ in momentum representation:

$$
\langle\mathbf{k}'|v^{(1)}|\mathbf{k}\rangle = \int \Omega_{fi}(\mathbf{p}',\mathbf{p})\langle\mathbf{k}',\mathbf{p}'|t(E)|\mathbf{k},\mathbf{p}\rangle\frac{d\mathbf{p}}{2\pi^3}\frac{d\mathbf{p}'}{2\pi^3} \qquad (5.5)
$$

The above expression can be further simplified if we assume that the total momentum is conserved. Conservation of momentum is an approximation because the residual nucleus may take some momentum.

[1] Reminder: $\tilde{\Psi}(\mathbf{k}')and\Psi(\mathbf{k})$ are the optical wave-functions used in the expansion of the Green's function for a P_1 subspace Equation (2.16)

$$\langle \mathbf{k}', \mathbf{p}' | t(E) | \mathbf{k}, \mathbf{p} \rangle = (2\pi^3)\delta(\mathbf{k}' + \mathbf{p}' - \mathbf{k} - \mathbf{p}) t(\mathbf{k}', \mathbf{k}; \mathbf{p}; E). \tag{5.6}$$

Of course, the above expression is only valid for a scattering in which the target nucleon remains bound after scattering with the projectile. If a bound nucleon is knocked-out after scattering with the projectile, the above equations will be modified in the same way the FKK theory was modified to describe a two-particle final state. We have shown in Chapter 2 that the cross section for a knockout-process in the FKK formalism is a convolution of the terms of the form $\langle \tilde{\Psi}(\mathbf{k}')^{(+)} \tilde{\Psi}(\mathbf{p}')^{(+)} | v | \Psi(\mathbf{k})^{(+)} \rangle$ or $\langle \mathbf{k}', \mathbf{p}' | v | \mathbf{k} \rangle$ in a momentum representation:

$$\langle \mathbf{k}', \mathbf{p}' | v^{(1)} | \mathbf{k} \rangle = \int \frac{d\mathbf{p}}{2\pi^3} \Omega_{fi}(\mathbf{p}) \langle \mathbf{k}', \mathbf{p}' | t(E) | \mathbf{k}, \mathbf{p} \rangle \tag{5.7}$$

$$\langle \mathbf{k}', \mathbf{p}' | t(E) | \mathbf{k}, \mathbf{p} \rangle = \delta(\mathbf{k}' + \mathbf{p}' - \mathbf{k} - \mathbf{p}) t(\mathbf{k}', \mathbf{k}; \mathbf{p}; E) \tag{5.8}$$

where

$$\Omega_{fi}(\mathbf{p}) = \int \frac{d\mathbf{p}_1}{2\pi^3}\frac{d\mathbf{p}_2}{2\pi^3}\cdots\frac{d\mathbf{p}_{j-1}}{2\pi^3}\frac{d\mathbf{p}_{j+1}}{2\pi^3}\cdots\frac{d\mathbf{p}_A}{2\pi^3} \tag{5.9}$$

$$\cdot\Phi^{\dagger}_{\rm lh}(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{j-1},\mathbf{p}_{j+1}\cdots\mathbf{p}_A) \tag{5.10}$$

$$\cdot\Phi_0(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{j-1},\mathbf{p},\mathbf{p}_{j+1}\cdots\mathbf{p}_A) \tag{5.11}$$

5.3 Summary

We have described a simple prescription for evaluating the optical potential (v) matrix elements in terms of nucleon-nucleon t-matrix elements. This prescription raises the upper energy limit of applicability of the MSD scattering formalism because the multiple t-matrix scattering describes nuclear data very well in the energy region above that of FKK applicability [22]. We have evaluated the optical potential matrix elements to the first order only, although it is straightforward to carry the computation to the second and the third order optical potentials which have been evaluated in terms of multiple t-matrix scattering.

Appendix A

Factorization of Green Function using the Chaining Hypothesis

In this section we will show how the full propagator $1/(E^{(+)} - H_{\rm opt})$ is factorized into P-chain propagators, each one representing a step in the P-chain. We consider the full propagator projected by P_μ from the left and P_i from the right:

$$\begin{aligned} P_\mu \frac{1}{E^{(+)} - H_{\rm opt}} P_i &\equiv P_\mu \mathcal{G} P_i = \mathcal{G}_\mu \\ H_{\rm opt} &= H^{(D)} + v \end{aligned} \tag{A.1}$$

First we multiply $\mathcal{G} \equiv 1/(E^{(+)} - H_{\rm opt})$ by $E^{(+)} - H^{(D)} - v$ to get

$$(E^{(+)} - H^{(D)} - v)\mathcal{G} = 1 \tag{A.2}$$

Multiplying by P_μ from the left and P_i from the right and inserting the full set of projectors $\sum_\nu P_\nu = 1$ in between we get:

$$P_\mu (E^{(+)} - H^{(D)} - v) \sum_\nu P_\nu \mathcal{G} P_i = 0 \tag{A.3}$$

Next we employ the chaining hypothesis which says that $P_\mu v P_\nu \neq 0$ only when $\nu = \mu, \mu \pm 1$ to get

$$(E^{(+)} - H^{(D)})\mathcal{G}_\mu = v_{\mu,\mu-1}\mathcal{G}_{\mu-1} + v_{\mu,\mu+1}\mathcal{G}_{\mu+1} \tag{A.4}$$

We start our factorization P_M which is sufficiently complex that the transition to the more complex spaces can be neglected, i.e. $P_{M+1}vP_M = 0$. Then the above equation reads:

$$(E^{(+)} - H^{(D)})\mathcal{G}_M = v_{M,M-1}\mathcal{G}_{M-1} \tag{A.5}$$

from which follows

$$\mathcal{G}_M = G_M v_{M,M-1}\mathcal{G}_{M-1} \tag{A.6}$$

where

$$G_M = \frac{1}{E^{(+)} - H^{(D)}} \tag{A.7}$$

Employing equation (A.4) for $\mu = M - 1$ we get

$$(E^{(+)} - H^{(D)})\mathcal{G}_{M-1} = v_{M-1,M-2}\mathcal{G}_{M-2} + v_{M-1,M}\mathcal{G}_M \tag{A.8}$$

and employing equation (A.6) we get:

$$(E^{(+)} - H^{(D)} - v_{M-1,M}G_M v_{M,M-1})\mathcal{G}_{M-1} = v_{M-1,M-2}\mathcal{G}_{M-2} \tag{A.9}$$

or

$$\mathcal{G}_{M-1} = G_M v_{M-1,M-2}\mathcal{G}_{M-2} \tag{A.10}$$

where

$$G_M = \frac{1}{E^{(+)} - H^{(D)} - v_{M-1,M}G_M v_{M,M-1}} \tag{A.11}$$

By mathematical induction we get

$$\mathcal{G}_\mu = G_\mu v_{\mu,\mu-1}\mathcal{G}_{\mu-1} \tag{A.12}$$

where

$$G_\mu = \frac{1}{E^{(+)} - H^{(D)} - v_{\mu-1,\mu}G_\mu v_{\mu,\mu-1}} \tag{A.13}$$

By reiterating (A.12) and replacing P_i by P_ν for generality we get the form of the factorized propagator we will employ frequently:

$$\begin{aligned}
\mathcal{G}_{P_\mu P_\nu} &= P_\mu \frac{1}{E^{(+)} - H_{\text{opt}}} P_\nu \\
&= G_{P_\mu} v_{P_\mu,P_{\mu-1}} \mathcal{G}_{P_{\mu-1}P_\nu} \\
&= G_{P_\mu} v_{P_\mu,P_{\mu-1}} G_{P_{\mu-1}} v_{P_{\mu-1},P_{\mu-2}} \cdots G_{P_{\nu+2}} v_{P_{\nu+2},P_{\nu+1}} G_{P_\nu}
\end{aligned} \tag{A.14}$$

Appendix B

Derivation of multiplicative constant in the final expression for the FKK MSD cross section (2.34)

In this appendix it will be explained how multiplicative constants and state densities are grouped together to obtain the final expression for the MSD cross section (2.34).

The multiplicative constants and state densities that enter the expression for the MSD cross section (2.34) are:

1. $\left[\frac{2\pi}{\hbar}\frac{m}{\hbar k_i}\rho(k_f)\right]$ from the expression for the MSD cross section

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}\frac{m}{\hbar k_i}\rho(k_f)\sum_{\mu}|\mathcal{T}_{fi}^{(\mu)}|^2 \tag{B.1}$$

2. $\left[\frac{1}{2\pi^2\,\rho(k_f)}\right]$ from the final term in the MSD convolution expression for $|\mathcal{T}_{fi}^{(\mu)}|^2$ (2.31).

$$|\mathcal{T}_{fi}^{(\mu)}|^2 = \int \frac{d\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d\mathbf{k}_\mu}{(2\pi)^3} \left[\frac{1}{2\pi^2\,\rho(k_f)}\right] \left[\frac{d^2w^{(1)}_{\mu+1,\mu}(\mathbf{k}_f \leftarrow \mathbf{k}_\mu)}{dU_f d\Omega_f}\right] \times \left[\frac{d^2\tilde{w}^{(1)}_{\mu,\mu-1}(\mathbf{k}_\mu \leftarrow \mathbf{k}_{\mu-1})}{dU_\mu d\Omega_\mu}\right] \cdots \left[\frac{d^2\tilde{w}^{(1)}_{1,i}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right] \tag{B.2}$$

This factor appears because integration is carried over intermediate momenta $\mathbf{k}_1, \cdots, \mathbf{k}_\mu$ but not over $\mathbf{k}_f$. It can now be easily seen that when $|\mathcal{T}_{fi}^{(\mu)}|^2$ in (B.2) is inserted in the expression for the MSD cross section (B.1) the density of states $\rho(k_f)$ is cancelled.

We then multiply terms described in 1. and 2. to obtain $\frac{m}{\pi\hbar^2 k_i}$. This constant is then grouped with the initial amplitude in the convolution $\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dE_1 d\Omega_1}$ converting it into a cross section.

$$\frac{d^2\tilde{\sigma}^{(1)}(E_1, \Omega_1 \leftarrow E_i, \Omega_0)}{dE_1 d\Omega_1} = \frac{m}{\pi\hbar^2 k_i} \frac{d^2\tilde{w}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dE_1 d\Omega_1} \tag{B.3}$$

Even though $\frac{d^2\tilde{\sigma}^{(1)}(E_1,\Omega_1 \leftarrow E_i,\Omega_0)}{dE_1 d\Omega_1}$ above it has units of cross section, the matrix-elements are not DWBA as is indicated by placing $\tilde{}$ over σ. Only the last amplitude in the convolution, $\frac{d^2w^{(1)}_{\mu+1,\mu}(\mathbf{k}_f \leftarrow \mathbf{k}_\mu)}{dU_f d\Omega_f}$, has DWBA matrix elements even though it does not have units of cross section.

The above relationship between a single step direct cross section and a single step scattering amplitude (B.3) can be used to express the MSD cross section as a convolution of cross sections rather than amplitudes. The transition from amplitudes to cross sections is generally accompanied by changing the convolution integration variables from momenta to energies and angles. Consequently there are two factors to consider here: $[\frac{\pi\hbar^2 k_1}{m}]$ coming from conversion of single step amplitude to cross section, and $[\frac{1}{(2\pi)^3}\frac{mk_1}{\hbar^2}\,dE_1 d\Omega_1]$ which replaces $\frac{d\mathbf{k}_1}{(2\pi)^3}$. Multiplying these factors together and utilizing $\hbar^2 k_1^2 = 2mE_1$ to express momentum k_1 in terms of E_1 yields $\frac{mE_1}{4\pi^2\hbar^2}$ as the total conversion factor, so that

$$\int \frac{d\mathbf{k}_1}{(2\pi)^3}\left[\frac{d^2\tilde{w}^{(1)}(\mathbf{k}_2 \leftarrow \mathbf{k}_1)}{dU_2 d\Omega_2}\right]\left[\frac{d^2\tilde{\sigma}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right] = \frac{m}{4\pi^2\hbar^2}\int E_1 dE_1 d\Omega_1 \left[\frac{d^2\tilde{\sigma}^{(1)}(\mathbf{k}_2 \leftarrow \mathbf{k}_1)}{dU_2 d\Omega_2}\right]\left[\frac{d^2\tilde{\sigma}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right] \quad \text{(B.4)}$$

Of course this factor can be inserted for every integration over intermediate variables so that the MSD cross section can be written as a convolution of single-step cross sections:

$$\frac{d^2\sigma^{(\mu)}(E_f, \Omega_f \leftarrow E_0, \Omega_0)}{d\Omega_f dE_f} = \left(\frac{m}{4\pi^2\hbar^2}\right)^{\mu} \int E_\mu dE_\mu d\Omega_\mu \cdots \int E_1 dE_1 d\Omega_1 \times \left[\frac{d^2\sigma^{(1)}(\mathbf{k}_f \leftarrow \mathbf{k}_\mu)}{dU_f d\Omega_f}\right]\left[\frac{d^2\tilde{\sigma}^{(1)}(\mathbf{k}_\mu \leftarrow \mathbf{k}_{\mu-1})}{dU_\mu d\Omega_\mu}\right]\cdots\left[\frac{d^2\tilde{\sigma}^{(1)}(\mathbf{k}_1 \leftarrow \mathbf{k}_i)}{dU_1 d\Omega_1}\right] \quad \text{(B.5)}$$

The above expression can be written in a recursive fashion, as stated in (2.39),

$$\frac{d^2\sigma^{(\nu)}(E, \Omega \leftarrow E_0, \Omega_0)}{d\Omega dE} = \frac{m}{4\pi^2\hbar^2}\int d\Omega_{\nu-1}\int dE_{\nu-1}E_{\nu-1} \quad \text{(B.6)}$$

$$\times \frac{d^2\sigma^{(1)}(E, \Omega \leftarrow E_{\nu-1}, \Omega_{\nu-1})}{d\Omega dE}\frac{d^2\tilde{\sigma}^{(\nu-1)}(E_{\nu-1}, \Omega_{\nu-1} \leftarrow E_0, \Omega_0)}{d\Omega_{\nu-1}dE_{\nu-1}} \quad \text{(B.7)}$$

Appendix C

Density of States

C.1 Two-particle Density of States

In this appendix we show how the density of states of a two continuum particle system can be expressed in terms of single-particle densities of particles comprising the system. The most important ingredient in the derivation of our result is that the energy of each of the continuum particles is fixed, one at E_k and the other one at E_K. In order to express the number of two particle states in an energy interval $\rho_{2p}(E_k, E_K)\triangle E$ in terms of $\rho(E_k)\triangle E$ and $\rho(E_K)\triangle E$ to the lowest order in $\triangle E$ we use the fact that energy $\triangle E$ is distributed between the two particle in an arbitrary way:

$$\rho_{2p}(E_k, E_K)\triangle E = \int_{E_k}^{E_k+\triangle E} dE' \int_{E_K}^{E_k+E_K+\triangle E-E'} dE''\ \rho(E')\ \rho(E'') \qquad \text{(C.1)}$$

If the energy interval $\triangle E$ is small enough, so that ρ_1 does not vary appreciably over it then in the region of integration specified in the above equation, we can approximate $\rho(E')$ and $\rho(E'')$ by $\rho(E_k)$ and $\rho(E_K)$, respectively. Thus we obtain:

$$\rho_{2p}(E_k, E_K)\triangle E = \rho(E_k)\,\rho(E_K)\int_{E_k}^{E_k+\triangle E} dE' \int_{E_K}^{E_k+E_K+\triangle E-E'} dE'' \qquad \text{(C.2)}$$

$$= \rho(E_k)\,\rho(E_K)\int_{E_k}^{E_k+\triangle E} dE'(E_k+\triangle E-E')$$

$$= \rho(E_k)\,\rho(E_K)\{(E_k+\triangle E)\triangle E - \frac{1}{2}[(E_k+\triangle E)^2 - E_k^2]\}$$

$$= \frac{1}{2}\rho(E_k)\,\rho(E_K)(\triangle E)^2 \qquad \text{(C.3)}$$

Bibliography

[1] H. Feshbach, A. Kerman, and S. Koonin, Ann. Phys. (N.Y.) **125**, 429 (1980).

[2] R. Bonetti, M.B. Chadwick, P.E. Hodgson, B.V. Carlson, and M.S. Hussein, Physics Reports **202(4)**, 171 (1991).

[3] P.E. Hodgson and M.B. Chadwick, in *International Conference on Nuclear Data for Science and Technology*, Gatlinburg, Tennessee, May 9-13 1994, edited by K. Dickens (American Nuclear Society, in press).

[4] H. Nishioka, H.A. Weidenmuller, and S. Yoshida, Z. Phys. **A336**, 197 (1990).

[5] M.B. Chadwick, *D. Phil thesis*, Oxford University (1989).

[6] M.B. Chadwick and P.G. Young, Phys. Rev. C **47**, 2255 (1993).

[7] A. Marcinkowski, J. Rapaport, R.W. Finlay, X. Aslanaglou, Nucl. Phys. **A530**, 75 (1991)

[8] A. Marcinkowski, J.Rapaport, R.W. Finlay, C. Brient, M. Herman, and M.B. Chadwick, Nucl. Phys. **A561**, 387 (1993).

[9] A.J. Koning, O. Bersillon, and J.-P. Delaroche, in in *International Conference on Nuclear Data for Science and Technology*, Gatlinburg, Tennessee, May 9-13 1994, edited by K. Dickens (American Nuclear Society, in press); Y. Watanabe, A. Aoto, H. Kashimoto, S. Chiba, T. Fukahori, K. Hasegawa, M. Mizumoto, S. Meigo, M. Sugimoto, Y. Yamanouti, N. Koori, M.B. Chadwick, and P.E. Hodgson, in *ibid*, and to be submitted to Phys. Rev. C.

[10] P. Demetriou, P. Kanjanarat, and P.E. Hodgson, J. Phys. **G 19**, L193 (1993).

[11] K. Sato and S. Yoshida, Phys. Rev. C **49**,1099 (1994).

[12] M. Kawai, A. Kerman, and K.W. McVoy, Ann. Phys. (N.Y.) **75**, 156 (1973).

[13] P.G. Young, E.D. Arthur, and M.B. Chadwick, *Los Alamos National Laboratory Report* LA-12343-MS (1992).

[14] A.J. Koning and J.M. Akkermans, Ann. Phys. **208**, 216 (1991).

[15] A. Pavlik, A. Priller and H. Vonach, in *Proceedings of the Final Research Coordination Meeting (RCM-3) of the Co-ordinated Research Programme (CRP) on Measurements and Analysis of 14 MeV Neutron-Induced Double Differential Neutron Emission Cross Sections needed for Fission and Fusion Reactor Technology*, ed. Wang DaHai, Chiang Mai, Thailand, 31 March - 2 April 1992. (International Atomic Energy Agency document INDC(NDS)-272, 1993)p. 25.

[16] H. Feshbach, Phys. Rev. C **48**, R2553 (1993); H. Feshbach, Ann. Phys. (N.Y.) **159**, 150 (1985).

[17] M.B. Chadwick, P.G. Young, and F.S. Dietrich, in *Specialist Meeting on Intermediate Energy Nuclear Data*, Paris, May 30 - June 1, 1994, edited by P. Nagel (Nuclear Energy Agency of the OECD, in print); M.B. Chadwick, P.G. Young, P. Oblozinsky, and A. Marcinkowski, Phys. Rev. C **49**, R2885 (1994); G. Arbanas, M.B. Chadwick, F.S. Dietrich, and A.K. Kerman, Lawrence Livermore National Laboratory document UCRL-JC-117171 (1994).

[18] G. Ciangaru, *Three-body approach to the multiple scattering direct reactions: Statistical theory of the continuum specturm* Phys. Rev. C **30**, 479 (1984)

[19] T.S.H. Lee, and G.A. Miller, *Color transparency and high-energy $(p, 2p)$ nuclear reactions* Phys. Rev. C **45**, 1863 (1992)

[20] J.V. Pilcher, A.A. Cowley, D.M. Whittal, and J.J. Lawrie *Protons of 200 MeV incident on ^{12}C. I. Conincident proton emission from the continuum* Phys. Rev. C **40**, 1937 (1989)

[21] G. Arbanas, M.B. Chadwick, F.S. Dietrich, and A.K. Kerman *Linking of direct and compound chains in multistep nuclear reactions* Phys. Rev. C Rapid Communications, (March 1995)

[22] A.K. Kerman, H. McManus, and R.M. Thaler *The Scattering of Fast Nucleons from Nuclei*, Annals of Physics **8**, 551 (1959)

[23] H. Feshbach, *Theoretical Nuclear Physics, Nuclear Reactions*, J. Wiley & Sons, Inc. (1992)

[24] F.C. Williams, Nucl. Phys. **A166**, 231 (1971)

[25] W. Dilg, W. Schantl, H. Vonach, and M. Uhl, Nucl. Phys. **A217**, 269 (1973)

[26] H. Gruppelaar, *IAEA Advisory Group Meeting on Basic and Applied Problems on Nuclear Level Densities* (Brookhaven National Laboratory, 1983)

[27] G.R. Satchler, *Direct nuclear reactions.* Oxford University Press, New York, 1983

[28] N. Austern, *Direct nuclear reaction theories.* John Wiley & Sons, New York, 1970

[29] T. Tamura, T. Udagawa and H. Lenske, it Multistep direct reaction analysis of continuum spectra in reaction induced by light ions, Phys. Rev. C **26**, no. 2, p. 379, (1982)

[30] H. Nishioka, H.A. Weidenmuller and S. Yoshida, *Statistical theroy of precompound reactijons: The multistep direct process*, Ann. Phys. (N.Y.), **183**, p. 166, (1988)

[31] A.J. Koning *Multi-step direct reactions*, PhD Thesis, (1992)

[32] A.J. Koning *Computational comparison of quantum-mechanical models for multistep direct reactions*, Phys. Rev. C **47**, no. 2 (1993)

[33] M.B. Chadwick, P.G. Young, D.C. George, and Y. Watanabe *Multiple preequilibrium emission in Feshbach-Kerman-Koonin analyses*, Phys. Rev. C **50**, no. 2 (1994)

[34] W. Hauser and H. Feshbach, Phys. Rev. **87**, 366 (1952)

[35] J.M. Akkermans and H. Gruppelaar, Z. Phys. A **300**, 345 (1981)

[36] M. Blann and H. Vonach, Phys. Rev. C **28** 1475 (1983)

[37] S.V. Forthsch, A.A. Cowley, J.J. Lawrie, J.V. Pilcher, F.D. Smit, and D.M. Whittal, Phys. Rev. C **48**,743 (1993)

[38] R. Bonetti, A.J. Koning, J.M. Akkermans, P.E. Hodgson, Physics Reports **247** (1994)

[39] A.K. Kerman, *Private Communications*

THESIS PROCESSING SLIP

FIXED FIELD: ill. _____ name _____

index _____ biblio _____

▶ COPIES: (Archives) Aero Dewey Eng Hum

Lindgren Music Rotch (Science)

TITLE VARIES: ▶☐ _____

_____

_____

NAME VARIES: ▶☐ _____

_____

IMPRINT: (COPYRIGHT) _____

▶ COLLATION: 107 l. _____

_____

▶ ADD. DEGREE: _____ ▶ DEPT.: _____

SUPERVISORS: _____

_____

_____ __ _____

_____ __ _____

_____ __ _____

_____ __ _____

_____ __ _____

_____ __ _____

_____ __ _____

NOTES:

cat'r: date:

page: 5175

▶ DEPT: Phy

▶ YEAR: 1995 ▶ DEGREE: Ph.D.

▶ NAME: ARBANAS, Goran

_____